

## Revisiting Angular Momentum Quantization in Bohr's Atomic Model

## Revisitando a Quantização do Momento Angular do Modelo de Bohr

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### Abstract

One of the most important developments of modern science was the emergence of the concept of energy quantization which forced its way into our previous classical understanding of matter in nature. Modern day view of matter constituents and present understanding of atomic structure basic to quantum chemistry arose in those first decades of last century. In this line, a key feature for the understanding of microscopic atomic structures and their stability was the concept of angular momentum quantization. To capture the beauty and the power of these concepts is therefore extremely relevant for apprehending the contemporary view of matter, energy and interactions in nature. In this work we present a possible alternative approach of quantizing the atomic orbit for the hydrogen's atom model proposed by Bohr without using his hypothesis of angular momentum quantization from the start. In our approach here we show that when we identify the number of photons and the quantum number for the orbit, we are led to Bohr's hypothesis as a consequence of the Planck's energy quantization.

**Keywords:** Atom, Quantization, Angular Momentum.

### Resumo

Um dos desenvolvimentos mais importantes da ciência moderna foi o surgimento do conceito de quantização da energia que mudou nossa compreensão clássica da natureza da matéria. A visão moderna dos constituintes da matéria e a compreensão atual da estrutura atômica básica para a química quântica surgiram naquelas primeiras décadas do século passado. Nessa linha, uma característica fundamental para o entendimento das estruturas atômicas microscópicas e sua estabilidade foi o conceito de quantização do momento angular. Da beleza e do poder desses conceitos é extremamente relevante apreender a visão contemporânea da matéria, energia e interações na natureza. Neste trabalho apresentamos uma abordagem alternativa de quantização da órbita atômica para o modelo do átomo de hidrogênio proposto por Bohr sem usar sua hipótese de quantização do momento angular. Em nossa abordagem mostramos que quando identificamos o número de fótons e o número quântico orbital, somos levados à hipótese de Bohr como consequência da quantização de energia de Planck.

**Palavras-chave:** Átomo, Quantização, Momento Angular.

## 1. Introduction

In terms of scientific endeavors, we can say that the XVIIth and XVIIIth centuries were characterized by the acquisition of a great number of information obtained experimentally both in Physics and in Chemistry. At the end of this period, many chemical facts emerged together with many questions, but few satisfactory explanations were known or presented. At the beginning of the XIXth century it was thought that there were fundamental differences among the chemical elements, its composites and mixtures. Mass conservation laws and definite composition laws had been proposed and sustained by some chemists, but none were able to explain satisfactorily why the masses were conserved and why the given materials have always the same composition. Therefore, in the idea of atomic composition and atomic models some explanations were sought for them.

In 1803, John Dalton (DALTON, 1808) proposed a non speculative but scientific theory that explained mass conservation and the constancy of composition of materials. These are Dalton's atomic model postulates:

- All matter is formed by atoms;
- Atoms are permanent and indivisible;
- Elements are characterized by their atoms (that is, atoms of the same element are equal and atoms of different elements are different, having different properties);
- Chemical transformations imply that atoms are combined and/or rearranged.

A first electrostatics model for the atom was proposed by the German physicist Wilhelm Eduard Weber as an aftermath of his research on electromagnetism. In 1862 (WEBER, 1862) and later on in 1871 (WEBER, 1871), he presented a modification to the “atom” proposed by another German physicist and philosopher Gustav Theodor Fechner. His proposed “atom” consisted of a central massive part that attracted gravitationally a cloud of almost imponderable or weightless particles. Weber then supposed that the central part of this “atom” was electrified with charge of some sign and that electric charges of opposite sign orbited around this central part according to the force law that he had proposed in 1846, in the first of his publications called *Elektrodynamische Maassbestimmungen* (“Electrodynamic measures”).

Beginning in 1880, the Dutch physicist Hendrik Antoon Lorentz started working on a new electrostatics model for the atom as all matter constituents, according to whom “electrons” (to him they meant any charged particle, positively or negatively, with or without mass) were distributed in the depths of the matter and free to oscillate with certain proper frequencies around fixed positions. He used this model to develop his Theory of Electron in 1892, and that permitted him explain dispersion of light and also the Zeeman effect. However, his model had difficulties in studying X-ray scattering.

To get around this difficulty, British physicist Sir Joseph John Thomson, in 1899 (THOMSON, 1899), started developing an atomic model considering it as a composite of a great number of negatively charged “corpuscles” (“corpuscles” which later on were recognized as electrons, for which he himself had determined the relation between their charge and mass in 1897) and “some” positive charge that balanced the total negative charge. Soon afterward, in 1904 Thomson (THOMSON, 1904), came up with a new model for the atom to explain that scattering. Let us see how.

Two years after the measuring of the ratio charge/mass for the electron (THOMSON, 1897), Thomson, in 1899 (THOMSON, 1899), started to elaborate a new model for the atom, considering it as a composite of a great number of electrons (in fact he did not use the word electrons, but corpuscles in a generic sense) and “some” positive charge that balanced the total negative charge. This vague idea about the positive charge in the atom was substituted in 1904 by the model in which the atom would be a spherical homogeneous distribution of positive charge in a radius of the size of order  $1 \text{ \AA}$  ( $10^{-8} \text{ cm}$ ). This concept that the positive electric charge would be distributed uniformly in the atomic volume had already been proposed by Lord Kelvin in 1902 (KELVIN, 1902). Within this positive charge distribution would be electrons uniformly distributed in concentric rings (THOMSON, 1904).

The dynamics and stability of the movement for these rings were treated in his article of 1904. Each electron with charge  $e$  and mass  $m$  was considered linked to the center of the atom and oscillating with a damping oscillation with proper frequency  $\omega_0$ . Such a configuration received the funny denomination of “plum pudding”. The name is inadequate for in a “pudding”, the number of plums is randomly distributed, whereas in Thomson's atomic model "electrons are uniformly distributed in concentric rings so as to satisfy the stability conditions that ensured the equilibrium, with a postulate that the number of these rings was minimum". With this model he tried to explain the then known properties of matter:

- visible light emission by bodies heated to high temperatures;
- X-ray emission of the same nature as visible light but much shorter wavelength, when cathodic rays hit a material obstacle;
- heavy atoms emission of three types of radiation,  $\alpha$ ,  $\beta$  and  $\gamma$ : it was the natural radioactivity discovered by Pierre and Marie Curie (CURIE, 1898).  $\alpha$  rays were identified in twice ionized helium atoms.  $\beta$  rays, with velocities greater than cathodic rays, are, like these, constituted by negative charged electrons.  $\gamma$  rays, similar to X-rays, are massless and much more penetrating.

In 1904, the Japanese physicist Hantaro Nagaoka (NAGAOKA, 1904) proposed an atomic model in which the atom was formed by a central part charged positively and surrounded by rings of electrons moving with the same angular velocity, a system similar to planet Saturn, with its rings, the reason why his model became known as the saturnian model. With this model, Nagaoka tried to explain the spectral lines, as well as radioactive emissions  $\alpha$  and  $\beta$ . With effect, to him, the perpendicular oscillations to the rings's plan of movement resulted in "band type" spectra (continuous), while the parallel oscillations to that plan resulted in "line type" spectra (discrete). On the other hand, the breakage of one of these rings provoked the  $\beta$  emission (decay). It is interesting to make stand out that a first idea of this type of "saturnian" model had already been presented in 1901 (CARUSO; OGURI, 2006), by the French physicist Jean Baptiste Perrin, as he considered the hypothesis that the electrons in atoms moved along orbits around a central lump with velocity of the same order as the velocities by which electrons are ejected from the surface of aluminum (Al) in a photoelectric effect. If such occurred, Perrin observed, the frequency of revolution of electrons was of the order of the optical frequencies of spectrum lines. Yet for Perrin, the instabilities of electronic orbits in his model were responsible for radioactive phenomena and, specially, for  $\beta$  emission.

One of the great difficulties faced by both Thomson's and Nagaoka's models was to know the number of electrons in each ring, and also, to explain its stability in virtue of Larmor's radiation. Before these models being formally presented, it already existed a difficulty in knowing the number of electrons and their respective distribution in the atom's interior.

However, the greatest difficulty of Thomson's model appeared when Rutherford and his collaborators, the German physicist Hans (Joahannes) Wilhelm Geiger and the British physicist Ernst Marsden, started studying  $\alpha$  particle scattering by matter. With effect, in 1906 (RUTHERFORD, 1906), Rutherford presented the results of experiments in which it was observed a small scattering (approximately 2 degree deviation) of  $\alpha$  particles when passing through a gold plate with 0,003 cm thickness. In 1908 (Proceedings of the Royal Society of London A81, p. 174), Geiger studied the scattering of an  $\alpha$  particle beam from a radium compound, namely, radium bromide ( $\text{RaBr}_2$ ) through a slim plate of metal (Aluminum, Al, and gold, Au). Scattered  $\alpha$  particles were detected in scintillation counters. Using this counting technique, Geiger and Marsden, in 1909 (GEIGER; MARSDEN, 1909), studied the scattering of a beam of  $\alpha$  particles originated in radon (Rn), through a slim metal plate. In this study, they observed that of the beam, not very well collimated and containing about 8,000 such particles, only one was reflected, that is, was scattered in an angle greater than 90 degrees ( $> 90^\circ$ ). This type of experiment was also commented upon by Geiger in 1910 (GEIGER, 1910). Still in 1910 (THOMSON, 1910), Thomson himself has shown that his model did not explain the results obtained by Geiger and Marsden.

In 1911 Rutherford (RUTHERFORD, 1911) interpreted the results of Geiger and Marsden, proposing his famous planetary model for the atom. Notwithstanding this Rutherford's planetary

formulation, Nagaoka's saturnian model still was employed by the British physicist John William Nicholson (1881-1955) in his research on cosmic spectral lines. With effect, in 1911 (Monthly Notices of the Royal Astronomical Society 72, pgs. 49; 139) and in 1912 (Monthly Notices of the Royal Astronomical Society 72, pgs. 677; 693; 729), he developed a new saturnian atomic model. However, in order to deter Larmor's radiation due to the movement of electrons in their rings, Nicholson considered null the vectorial sum of these electrons's acceleration, and that their angular momenta should vary discretely and in quantities proportional to the Planck's constant. Be it registered that this hypothesis was employed by the Danish physicist Niels Henrik David Bohr, in 1913. With this new saturnian model, Nicholson explained that the spectral lines were due to small vibrations of electronic rings of the primary atoms, which according to his understanding, were of three types: coronium, containing two electrons; hydrogen, with three electrons and nebulium, with four electrons. To Nicholson, helium was considered a composite element. Later, it was shown that the nebulium was nothing more than a metastable mixture of oxygen (O) and nitrogen (N), and that coronium was highly ionized iron (Fe).

With the experimental confirmation for the existence of a nucleus, the problem of the atom's stability gets worse. In a simplified vision, if the electrons circulated around the nucleus, they would be constantly accelerated and, therefore, they would loose energy by electromagnetic radiation emission, in such a way that their orbital radius would diminish until colliding with the nucleus. Then one could conclude that such atom would emit a continuous spectrum, which does not agree with data from spectroscopy.

The difficulties of the classical atomic models discussed in this section were solved by the quantum atomic model, developed since the work of Bohr (BOHR, 1913), started in 1913, model nowadays known as quantum atomic model of Bohr-Ishiwara-Wilson-Sommerfeld or the old quantum mechanics.

In this historic development (BASSALO, 1987; CARUSO, 2006) of the atomic model building, the quantization of the angular momentum was imposed to ensure not only the stability condition for the orbit but to avoid the classical collapsing of the electron onto its nucleus. In our work here we start also with the stability condition for the atomic orbital and require that the stable minimum for the Planck's energy quantization to arrive at the quantization for the angular momentum.

The outline for this work is as follows: First section is dedicated to reviewing the basic assumptions in the Bohr's atomic model which contains the seminal ideas for quantizing the atomic orbitals. Next we retrace the historical context in which his successful model arose, both from theoretical as well as experimental contributions of his contemporary colleagues. Next we introduce and develop our main ideas concerning the requirement to minimize Planck's energy and use it to identify with Bohr's quantization hypothesis, and finally the last section is for our main conclusions.

## 2. Bohr's Hypothesis

It was then that Bohr's crucial contribution came in. Analysis of the hydrogen spectrum which showed that only light at certain definite frequencies and energies were emitted led him to postulate that the circular orbit of the electron around the nucleus is quantized, that is, that its angular momentum could only have certain discrete values, these being integer multiples of a certain basic value. This was his "*ad hoc*" assumption, introduced by hand into the theory. In 1913, therefore, he proposed the following for the atomic model (BOHR, 1913):

1. The atom would be composed of a central nucleus where the positive charges (now known to be carried by protons) are located;
2. Around the central nucleus revolved the electrons in equal number as the positive charges present in the nucleus. The electrons orbiting such a nucleus had discrete quantized energies, which meant that not any orbit is allowed but only certain specific ones satisfying the energy quantization requirements;

3. The allowed orbits would also have quantized or discrete values for orbital angular momentum, according to the prescription  $|L| = n\hbar$  where  $\hbar = h/2\pi$  and  $n = 1, 2, 3, \dots$ , which meant the electron's orbit would have specific minimum radius, corresponding to the angular momentum quantum number  $n = 1$ . That would solve the problem of collapsing electrons into the nucleus.

Observe that Bohr did not use the value  $n = 0$  because this does not define an electronic orbit around the nucleus, although in Planck's hypothesis it is perfectly allowed. The reason why Bohr left out this first quantum number out of his hypothesis comes from experiment, since spectroscopic studies of many chemical elements show that these numbers start with  $n = 1$ .

Two corollaries from Bohr's assumptions do follow: First, from item 2. above, the laws of classical mechanics cannot describe the transition of an electron from one orbit to another, and second, when electrons do make a transition from one orbit to another, the energy difference is either supplied (transition from lower to higher energy orbits) or carried away (transition from higher to lower energy orbits) in discrete values only. Today, we ascribe to the photon the role as carrier of such a quantum of energy.

Let us then now briefly review the usual pathway where Bohr's quantization is introduced. Using Newton's second law for the electron charge  $e$ , mass  $m_e$  and moving in a circular orbit, radius  $r$ , around the nucleus, and thus subject to Coulomb's law, we have (taking  $Z = 1$  for simplicity):

$$\frac{e^2}{4\pi\epsilon_0 r^2} = m_e \frac{v^2}{r} \quad (1)$$

This allows us to calculate the kinetic energy  $E_k$  of the electron in such an orbit:

$$E_k = \frac{1}{2} m_e v^2 = \frac{e^2}{8\pi\epsilon_0 r} \quad (2)$$

The potential energy  $E_p$  for the same system on the other hand is given by

$$E_p = -\frac{e^2}{4\pi\epsilon_0 r} \quad (3)$$

Therefore, the total energy  $E$  for the system is

$$E = -\frac{e^2}{8\pi\epsilon_0 r} \quad (4)$$

This result would suggest that, since the radius can have any value, the same should happen with the angular momentum  $L$ .

$$L = p r \sin \theta = p r, \text{ for } \theta = 90^\circ \quad (5)$$

that is, the angular momentum depends on the radius  $r$ . The linear momentum of the electron is given by

$$p = m_e v. \quad (6)$$

Therefore, the problem of quantizing the angular momentum  $L$  reduces to the quantizing of the radius  $r$ , which depends on the total energy (4). Just here Bohr introduced an additional hypothesis, in that the angular momentum of the electron is quantized, i.e.,

$$L = n\hbar, \quad (7)$$

In this manner he was able to quantize the other physical quantities such as the total energy. This is the usual pathway wherein the textbooks normally follow in their sequence of calculations.

### 3. Historical sketch on angular momentum quantization

Angular momentum quantization has its origins in the research on atomic and molecular spectroscopy done between 1910 and 1913. It was in this context that the idea of angular momentum quantization emerged. We believe that this quantization had its beginning in the search of relating Planck's constant to the structure of matter.

The first one that tried to relate Planck's constant to the atomic constituents was Arthur Erich Haas (HAAS, 1910) using Thomson's atomic model. Some researchers in the history of physics state that Haas had been influenced by Einstein's ideas that there should be a way to associate two facts that could not be accounted for by Maxwell's classical electrodynamics: the quantum nature of radiation (EINSTEIN, 1905) and the existence of electrons, viz the quantum structure of matter.

Haas assumed the hypothesis that electrons move around the most external orbit of Thomson's model, which corresponds to radius  $r$  of positive charge distribution. We observe that the geometry that corresponds to a spherical symmetry in Thomson's model and the Coulomb's force, implies a similar result if we used Rutherford's model for a sole electron. Following Planck's argument, Haas considered real atoms as ideal harmonic oscillators so that he could use the Planck's quantization rule for energy. So Haas got the atomic radius

$$r = \frac{\hbar^2}{2e^2m} \quad (8)$$

which is nowadays known as Bohr's radius for the ground state of an atom. This equation is valid only for the fundamental state of an atom, not applicable to excited states.

It is important to emphasize that other attempts were made trying to associate Planck's constant in atomic physics before Bohr's work which had relative success, but had to be abandoned when Rutherford evidenced the limitations of Thomson's model.

It was Nicholson who adopting the then recent atomic model proposed by Rutherford postulated that the angular momentum of an atom could increase or decrease by discrete amounts (NICHOLSON, 1912). Nicholson applied his ideas to study the spectrum of nebulae, accepting the idea that the spectrum had its origin in different atoms, defining what it is called different states of atoms, characterized by its internal movements.

However, Nicholson's ideas did not lead to an explanation of Balmer's formula. It was Bohr whom correctly interpreted the significance and importance of these spectral lines of atomic elements, postulating the angular momentum quantization.

The first successful application of quantum principles was in 1912 by chemist Niels Bjerrum (BJERRUM, 1912) within the domain of molecular spectroscopy and not atomic spectroscopy. Following suggestions from Lorentz, he supposed that the line that unites two atoms of a molecule (diatomic molecule) rotates on a plan and admitted, inspired in Nernst (NERNST, 1911) that the rotational energy would be a multiple of  $hf$ , where  $f$  is the number of rotations per second performed by the molecule.

Bjerrum thus arrived at:

$$\frac{1}{2}I\omega^2 = \frac{1}{2}I(2\pi f)^2 = nhf, (n = 1, 2, 3 \dots) \quad (9)$$

Therefore, for the frequency we have

$$f_n = \frac{nh}{2\pi^2 I} \quad (10)$$

In this way, we see that frequency is quantized and  $I$  is the momentum of inertia. This is almost the angular momentum quantization except for a factor of 2.

Em 1913, Ehrenfest (EHRENFEST, 1913) showed that the energy of rotation is purely kinetic and consequently the quantized rotational energy must equal  $(1/2)hf$  and not  $hf$  as postulated by Bjerrum (9). With this we have:

$$\frac{1}{2}I\omega^2 = \frac{1}{2}I(2\pi f)^2 = \frac{1}{2}nhf, (n = 1,2,3 \dots) \quad (11)$$

As  $I\omega$  is the moment of inertia for the molecule, it follows that,

$$L = n\hbar \quad (12)$$

Here we can see that Ehrenfest arrived at the angular momentum quantization for the molecule. Bohr's successful explanation for Balmer series of atomic spectra emerged in such an environment.

In our next section we present an alternative deduction for Bohr's angular momentum quantization using energy conservation and Planck's energy quantization.

#### 4. Orbital quantization from another perspective

Considering for simplicity an electron orbiting an atomic nucleus, the kinetic energy of the electron in such an orbit is given by (2) while the potential energy for the same system on the other hand is given by (3), so that the total energy for the system is (4).

Therefore, the total energy of the system we are considering is:

$$E_{total} = \frac{m_e v^2}{2} \quad (13)$$

Note that we can accommodate either the plus or minus sign here, depending on where we put the ground zero potential energy reference for the electron.

On the other hand, knowing that the scalar orbital velocity is given by

$$v = 2\pi f r \quad (14)$$

where  $f$  is the orbital frequency, substituting this  $v$  into (13) we then have:

$$E_{total} = 2\pi^2 m_e r^2 f^2 \quad (15)$$

so the ratio of the system's total energy variation with respect to the frequency is

$$\left. \frac{dE_{total}}{df} \right|_{System} = 4\pi^2 m_e r^2 f = 2\pi L \quad (16)$$

where (14), (5) and (6), were used to write the last term on the right hand. Here the variation of the total energy with respect to the frequency is subscripted by "System" because it comes from the classical dynamics of the system.

Now, taking our view from a different perspective, we follow Planck considering the interaction of radiation (light) with matter, in which, according to Planck's quantization scheme is given by

$$E = nh\nu \quad (17)$$

where  $n$  is a natural number and  $\nu$  the frequency of the interacting radiation. As the interaction occurs, this energy varies with respect to its frequency according to:

$$\left. \frac{dE}{d\nu} \right|_{Planck} = nh. \quad (18)$$

It is conspicuous that we cannot equate (16) with (18) straightaway, because in the former equation  $f$  is the frequency of the orbital movement of the electron while in the latter,  $\nu$  is the frequency of the radiation, and they are, of course, different in principle.

Here comes our main assumption: Since the two energy variations, namely (16) and (18) are with respect to the atomic frequency  $f$  and radiation frequency  $\nu$  respectively, what would be the consequence of equating these frequencies so that we could equate their energy variations with respect to the frequency (that is, to allow us to equate 16 with 18).

So, assuming equality between frequencies, we have

$$\left. \frac{dE}{d\nu} \right|_{Planck} = \left. \frac{dE_{total}}{df} \right|_{System} \Leftrightarrow \text{if } \nu = f \quad (19)$$

from which finally follows

$$L = \frac{nh}{2\pi} = n\hbar \quad (20)$$

This is exactly Bohr's angular momentum quantization (7).

## 5. Conclusion

Looking from a different perspective, here we have deduced the angular momentum conservation for atomic orbitals firstly proposed and hypothesized by Bohr by making use of two principles: energy conservation and Planck's quantization for the radiation interacting with matter. Our approach differs from the traditional one where the quantization of the angular momentum for atomic orbitals is achieved by comparing the energy of the rotating electron with half of the energy of a quantum oscillator associated with it. In our approach here, energy conservation in the variation of the energy of electron's movement when compared to the variation of energy carried by the radiation interacting with the atom, which is quantized according to Planck's assumption, leads to angular momentum quantization à la Bohr.

In this work we have shown that Planck's fundamental assumption of energy quantization can be used to arrive at the Bohr's assumption of angular momentum quantization. In fact, we have shown that Bohr's rule for angular momentum quantization can be derived from Planck's energy quantization.

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## References

- Bassalo, J.M.F. (1987) Crônicas da Física 1, Editora Universidade Federal do Pará.
- Bohr, N. (1913). On the constitution of atom and molecules. Philosophical Magazine, S. 6, 26, n.151, p. 1-25 (1913). <http://dx.doi.org/10.1080/14786441308634955>
- Bjerrum, N. (1912). Infrared absorption spectra of gases. Nernst Festschrifte, p.90, Halle, Germany. [https://www.royalacademy.dk/Publications/Low/1686\\_Bjerrum,%20Niels.pdf](https://www.royalacademy.dk/Publications/Low/1686_Bjerrum,%20Niels.pdf)
- Caruso, F. & V. Oguri, (2006). Física Moderna: Origens Clássica e Fundamentos Quânticos, Editora Elsevier.
- Curie, S. (1898). Rayons émis par les compomses de l'uranium et du thorium. Comptes Rendus 126, p.1101-1103. [https://www.academiesciences.fr/archivage\\_site/activite/archive/dossiers/Curie/Curie\\_pdf/CR1898\\_p1101.pdf](https://www.academiesciences.fr/archivage_site/activite/archive/dossiers/Curie/Curie_pdf/CR1898_p1101.pdf)
- Dalton, J. (1808). A New System of Chemical Philosophy. Manchester. Edition, London. <https://www.paradigmaakademiyayinlari.com/wp-content/uploads/2021/06/john-dalton-and-the-rise-of-modern-chemistry.pdf>
- Ehrenfest, P. (1911). Welche züge der lichtquantenhypothese spielen in der theorie der wärmestrahlung eine wesentliche rolle? Annalen der Physik, 36, p. 91-118. <https://doi.org/10.1002/andp.19113411106>
- Ehrenfest, P. (1913). Berichte der Deutschen Physikalischen Gesellschaft. Ges. 15, 1159 (1913).
- Einstein, A. (1905). Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt. Annalen der Physik, Ser. 4, 17, p. 132-148. <https://doi.org/10.1002/andp.19053220607>
- Haas, A.E. (1910). Über eine neue theoretischer methode zur bestimmung des elektrischen elementar-quantums und des halbmessers des wasserstoffatoms. Physikalische Zeitschrift 11, n. 12, p. 537-538. <https://knowledge.electrochem.org/estir/hist/hist-12-Debye-1.pdf>
- Geiger, H. & Marsden, E. (1909). On a diffuse reflection of the  $\alpha$ -particles, Proceedings of the Royal Philosophical Society of London 82, p. 495-500. <https://doi.org/10.1098/rspa.1909.0054>
- Geiger, H. (1910). The scattering of  $\alpha$ -particles by matter. Proceedings of the Royal Society of London A83, p. 492-504. <https://doi.org/10.1098/rspa.1910.0038>
- Kelvin, Lord (William Thomson). (1902). Aepinus Atomized. Philosophical Magazine, S. 6, 3, n. 15, p. 257-283. <https://doi.org/10.1080/14786440209462764>
- Nagaoka, H. (1904). Kinetics of a system of particles illustrating the line and the band spectrum and the phenomena of radioactivity. Philosophical Magazine 7, p. 445-455. <https://doi.org/10.1080/14786440409463141>
- Nernst, W. (1911). Zur theorie der spezifischer wärme and über die anwendung der lehre von den energiequanten auf physikalisch-chemische fragen überhaupt. Zeistschrift für Elektrotechnik und Electrochemie 17, p. 265-275. <https://doi.org/10.1002/bbpc.19110171809>
- Nicholson, J.W. (1912). The constitution of the Solar Corona. Monthly Notices of the Royal Astronomical Society 72 p. 677-693.
- Rutherford, E. (1906). Retardation of the alfa-particle from radium passing trough matter, Phil. Mag. 12, p. 134-146. <http://dx.doi.org/10.1080/1478644060946352>
- Rutherford, E. (1911). The scattering of  $\alpha$  and  $\beta$  particles by matter and structure of the atom. Philosophical Magazine. 21, 669-688. <http://web.ihep.su/dbserv/compas/src/rutherford11/eng.pdf>
- Thomson, J.J. (1899). On the masses of the Ions in gases at Low Pressures. Philosophical Magazine 48, p. 547-567. <http://dx.doi.org/10.1080/14786449908621447>
- Thomson, J.J. (1897). Cathode Ray. Philosophical Magazine, S. 5, 44, p. 293-316. <http://dx.doi.org/10.1080/14786449708621070>
- Thomson, J.J. (1910). On the scattering of rapidly moving electrified particles, Cambridge Literary and Philosophical Society 15, part 5, p.456-467.

- Thomson, J.J. (1904). On the structure of the atom: an investigation of the stability and periods of oscillations of a number of corpuscles arranged at equal intervals around the circumference of a circle; with application of the results to the theory of atomic structure. *Philosophical Magazine*, S. 6, 7, p. 237-265 (1904). <https://doi.org/10.1080/14786440409463107>
- Weber, W. E. (1862). *Gesellschaft der Wissenschaften zu Göttingen, Mathematische Klasse* 10, p.3. <https://www.ifi.unicamp.br/~assis/Modelo-Planetario-de-Weber.pdf>
- Weber, W. E. (1871). *Gesellschaft der Wissenschaften, Mathematische-Physische Klasse* 10, p.1. <https://www.ifi.unicamp.br/~assis/Modelo-Planetario-de-Weber.pdf>