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# Unsteady Squeezing Flow and Heat Transfer Analysis of Magnetohydrodynamic Third-grade Nanofluid between Two Disks Embedded in a Porous Medium subjected to Thermal Radiation using Homotopy Perturbation Method

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## Abstract
The non-linear behaviours of non-Newtonian fluids under various flow conditions continue to arouse research interests in recent times. In this work, nonlinear analysis of unsteady squeezing flow and heat transfer of a third-grade nanofluid between two parallel disks embedded in a porous medium under the influences of thermal radiation and temperature jump boundary conditions is studied using homotopy perturbation method. The parametric studies from the series solutions show that for a suction parameter greater than zero, the lower disc's radial velocity increases while that of the upper disc decreases as a result of a corresponding increase in the viscosity of the fluid from the lower squeezing disc to the upper disc. An increasing magnetic field parameter and the radial velocity of the lower disc decrease while that of the upper disc increases. There is a recorded decrease in the fluid temperature profile as the Prandtl number increases due to a decrease in the third-grade fluid's thermal diffusivity. The results of this work can be used to advance the analysis and study of third-grade nanofluid flow behavior and heat transfer processes.

**Keywords:** Third-grade nanofluid. Squeezing flow. Magnetohydrodynamic. Thermal radiation. Temperature jump boundary conditions.

## Nomenclature

| | |
|---|---|
| A | Suction/injection parameter |
| B | magnetic field strength, Nm/A |
| $c_p$ | specific heat capacity at constant pressure, J/kgK |
| $c_w$ | specific heat capacity of the nanofluid, J/kgK |
| h | height of the flow, m |
| H | total distance between the two disks, m |
| $k_{nf}$ | thermal conductivity of the nanofluid, W/mK |
| $K_p$ | permeability of the porous medium, $m^2$ |
| $M$ | Hartmann number or magnetic field parameter |
| P | Pressure, $N/m^2$ |
| $\Pr$ | Prandtl's number |
| r | radius of the disk, m |
| **Sq** | Squeeze number |
| t | time, s |
| T | temperature, K |
| $\mu_f$ | dynamic viscosity of the basefluid, $Ns/m^2$ |
| $\mu_{nf}$ | dynamic viscosity of the nanofluid, $Ns/m^2$ |
| $\rho_{nf}$ | nanofluid density, $kg/m^3$ |
| $\alpha_{nf}$ | thermal diffusivity of the nanofluid, $m^2/s$ |
| $(\rho c_p)_{nf}$ | heat capacity of the nanofluid, $Jm^3/K$ |
| $\rho_f$ | density of the basefluid, $kg/m^3$ |
| $\theta$ | **d**imensionless temperature |
| $w$ | **a**xial velocity component, m/s |
| $u$ | **r**adial velocity component, m/s |
| $\eta$ | Similarity variable |
| $\gamma$ | **D**imensionless temperature jump parameter |
| $\sigma$ | Electrical conductivity, Ωm |
| $\upsilon$ | Kinematic viscosity, $m^2/s$ |
| $\alpha$ | Thermal diffusivity, $m^2/s$ |

## 1. Introduction

Squeezing flows of non-Newtonian fluids are evident in moving pistons, catalytic reactors, flow inside syringes and nasogastric tubes, oil recovery applications, chocolate fillers, power transmission, squeezed film, hydraulic lifts, compression, electric motors, injection modeling etc. The continuous wide areas of engineering, industrial and biological applications of fluid and heat transfer flow between two parallel discs or surfaces have inspired various researchers in recent times to critically study the problem. Considering the problem from heat and mass transfer analysis,

Mustafa et al. [1] performed transient heat transfer analysis of squeezing fluid flow through two parallel surfaces. Hayat et al. [2] did a study on second grade fluid being squeezed by two parallel discs and presented with proper comparison, the behaviour of the second-grade fluid considering or neglecting magnetic effect. In an attempt to solve the extension of Hayat et al. model considering suction and injection on magneto-hydrodynamic squeezing flow, Domairry and Aziz [3] used a semi-analytical method, Homotopy perturbation method (HPM) to obtain a symbolic solution for investigating and predicting the influences of suction and injection on magneto-hydrodynamic squeezing flow under standard conditions. A similar study was performed by Siddiqui et al. [4] using two parallel plates with the squeezing viscous fluid under transient condition.

Rashidi et al. [5] approached the problem using different analytical schemes. Considering the inclusion of nanotechnology into the problem of squeezing flow, Khan and Aziz [6-7], scrutinized the effect of natural convection on fluids with nanoparticles. They extended their work on squeezing nanofluids by later considering porosity. A further study on squeezing nanofluid flow between two plates was perform by Kuznestov and Nield [8]. They investigated the boundary layer of the squeezing process by using natural convection principle of fluid flow to generate different governing models. Hashimi et al. [9] found the models interesting and obtained analytical solutions to the governing equations with the assumption of zero slip and zero temperature jump. The researchers mentioned above performed their interesting research based on the assumption of zero slip and zero temperature jump. However, these two annulled assumptions have impacts on the processes in question. In other to provide an extension to the works of the above researchers as required, studies have been performed on squeezable fluids of different nano-particles sizes, different concentration, different stretching effect and different phases by taking slip into account [11-22]. The obtained models for the processes were documented and proper parametric studies were performed to understand the processes in general [23-34].

Past research works have presented magnetohydrodynamic fluid flow considering porous medium, nanofluid and some other factors capable of reshaping the squeezing process from idealization into actual [35-76]. In recent times, the flow and heat transfer characteristics of third grade nanofluid in pipes and in porous channel have been analyzed [77]. Hoseinzadeh et al. [78] performed a numerical validation on heat transfer through a porous fin with rectangular cross section that is subjected to laminar flow in an isotropic homogeneous medium. They employed an approximate analytical technique and verified the obtain result using numerical method. Thereafter, Darcy number was used for the parametric studies. In the same year, heat transfer though a porous fin with rectangular cross section is examined using Homotopy perturbation method (HPM) [79]. Furthermore, Hoseinzadeh et al. [80] analyzed thermal pulsating nanofluid flow over three different cross-sectional channels as well as an exploration of heat transfer of laminar and turbulent pulsating nanofluid flow using numerical methods [81]. Wang et al. [82] presented a three-dimensional methodology for visualizing natural convection in porous media. Wang et al. [83] also studied the impacts of porous fins on mixed convection and heat transfer mechanics in Lid-driven cavities. They modelled the process and performed parametric studies using the numerical solution obtained. Recently, Venkateswarlu and Bhaskar [84] considered MHD Casson fluid flow in a micro-channel with navier slip and convective boundary conditions. The work also presented the entropy generated as well as the dimensionless Bejan number. Additionally, Manjunatha et al. [85] investigated the impact of magnetic field, non-uniform temperature gradients and heat source on double diffusive Benard-Marangoni convection in a porous media. Since it has been established by Fosdick and Rajagopal [80] that third grade fluid gives different properties to those of fluids such as Newtonian and second grade fluids, Majhi and Nair [81] tried to manage the shearing stresses at the wall of a fluid flow process considering a third-grade fluid as the working fluid. They compared their obtained results to the numerical solutions by Massoudi and Christie [82] and an excellent agreement was reached by the two researchers. A third-grade fluid with constant viscosity was considered by Yurusoy and Pakdemirli [83]. In order to verify their work with the available numerical solution of Vajrevelu et al. [84] for third grade fluid flow, they considered a governing equation with varying viscosity under no slip condition. The research work gave interesting results and found great application in rotary devices.

Other studies on third grade fluids includes the fluctuating fluid flow by Hayat et al. [85], boundary layer analysis by Muhammet [86], heat transfer analysis by Yurusoy [87], entropy generation analysis by Pakdemirli et al. [88] and partial slip analysis by Sajid et al. [89]. Different schemes have also been employed to efficiently solve the resulting ordinary differential equations associated with squeezing fluid flow between two parallel surfaces [90- 96]. To the best of our knowledge, the study of magnetohydrodynamic squeezing unsteady flow of third grade nanofluid between two parallel disks embedded in a porous medium under the influences of thermal radiation and temperature jump condition has not been considered and analyzed in literature. Therefore, in this work, nonlinear analysis of unsteady squeezing flow and heat transfer of a third grade nanofluid between two parallel disks embedded in a porous medium under the influences of thermal radiation and temperature jump boundary conditions is studied analytically and numerically using homotopy perturbation method and the developed fifth-order Runge-Kutta Fehlberg method (Cash-Karp Runge-Kutta) coupled with shooting method. Homotopy perturbation method (HPM) is employed because it is independent of small parameters in the models and hence the restrictions of traditional perturbation can be eliminated. The obtained results make known that the method is effective, simple, accurate and can be applied to other nonlinear equations. Also, parametric studies are carried out and the influences of various parameters on the flow and heat transfer processes are established.

## 2. Problem Formulation

Consider an asymmetrical flow of third grade nanofluid through two parallel disks as shown in Fig. 1. The upper disk is moving towards the stationary lower disks under a uniform magnetic field strength applied perpendicular to disks as depicted in Fig. 1. The fluid conducts electrical energy as it flows unsteadily under the influence of magnetic force field. It is assumed that the fluid structure everywhere is in thermodynamic equilibrium and the plate is maintained at constant temperature

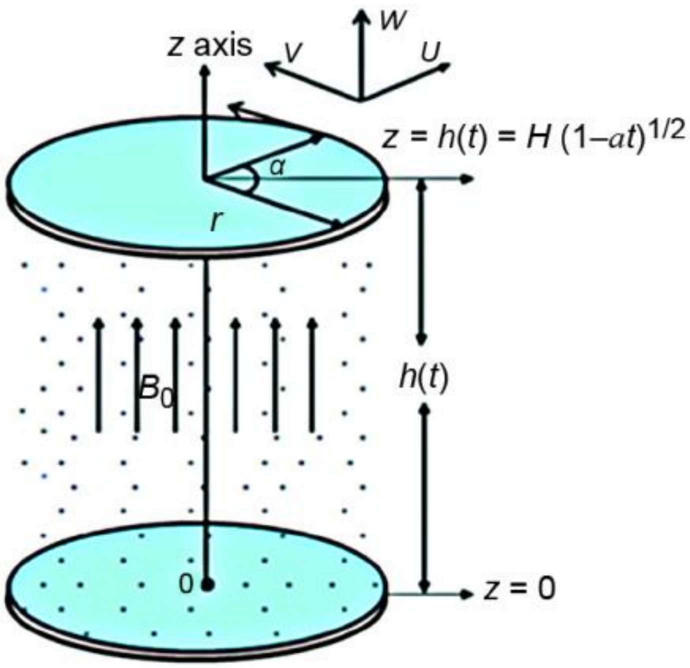


**Figure 1 – Problem formulation**

The Cauchy stress tensor, $\boldsymbol{\tau}$ for an incompressible homogeneous thermodynamically compatible third grade fluid is given by

$$\boldsymbol{\tau} = p\mathbf{I} + \mu\mathbf{A_1} + \alpha_1\mathbf{A_2} + \alpha_2\mathbf{A_1^2} + \beta_1\mathbf{A_3} + \beta_2\left(\mathbf{A_1A_2} + \mathbf{A_2A_1}\right) + \beta_3\left(\mathbf{trA_1^2}\right)\mathbf{A_1}. \tag{1}$$

Where $\boldsymbol{\tau}$ is the stress tensor, $\rho$ is the pressure, $\mathbf{I}$ is the identity tensor, $\mu$ is the dynamic viscosity, ***tr*** is the trace of a tensor, $\alpha_1, \alpha_2, \beta_1$, $\beta_2$ and $\beta_3$ are the material constant and $A_1$, $A_2$ and $A_3$ being the first, second and third Rivilin Erickson tensor. This Rivilin Erickson tensor is a temporal evolution of strain rate tensor such that the derivative rotate and translate with the flow field and can be derived from

$$\mathbf{A_1} = \left(grad\ \mathbf{V}\right) + \left(grad\ \mathbf{V}\right)^T, \tag{2}$$

$$\mathbf{A_n} = \frac{d\mathbf{A_{n-1}}}{dt} + \mathbf{A_{n-1}}\left(grad\ \mathbf{V}\right) + \left(grad\ \mathbf{V}\right)^T \mathbf{A_{n-1}}, \quad n \geq 1 \tag{3}$$

where **V** denotes velocity field, grad is the operator gradient and d/dt is the material time derivative. According to Fosdick Rajagopal [69] motion of the fluid must be in accordance with the thermodynamic model (Clausius Duhem inequality must be satisfied). When the motions of the fluid are thermodynamically compatible, the Clausius-Duhem inequality and the assumption that the Helmholtz free energy is minimum when the fluid is locally at rest (stable) require that

$$\mu \geq 0, \alpha_1 \geq 0, \beta_1 = \beta_2 = 0, \beta_3 \geq 0, \left|\alpha_1 + \alpha_2\right| \leq \sqrt{24\mu\beta_3}, \quad \beta_1 = \beta_2 = 0, \quad \beta \geq 0 \ . \tag{4}$$

Since $\beta_3 > 0$, the stress tensor can predict shear thickening as well as the normal stress.

Velocity and temperature fields are

$$\mathbf{V} = \left(u\left(t,r,z\right), 0, w\left(t,r,z\right)\right) \text{ and } \mathbf{T} = T\left(t,r,z\right). \tag{5}$$

Here, u and w are the radial and axial components of velocity. From the above Equation (1-5), the algebraic forms of the conservation equations can be developed with some extension on previous work [48] as;

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{6}$$

$$\rho_{nf}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} - \frac{\sigma_{nf} B_0^2 u}{1-ct} - \frac{\mu_{nf}}{K}u \tag{7}$$

$$\rho_{nf}\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rr}\right) + \frac{\partial \tau_{rz}}{\partial z} - \frac{\mu_{nf}}{K}w \ . \tag{8}$$

where

$$\begin{aligned}\tau_{rr} = &-p + 2\mu_{nf}\frac{\partial u}{\partial r} + 2\left(\alpha_1\right)_{nf}\left[u\frac{\partial^2 u}{\partial r^2} + w\frac{\partial^2 u}{\partial r \partial z} + 2\left(\frac{\partial u}{\partial r}\right)^2 + \frac{\partial w}{\partial r}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right] \\ &+ \left(\alpha_2\right)_{nf}\left[4\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2\right] + 4\left(\beta_3\right)_{nf}\frac{\partial u}{\partial r}\left[2\frac{u^2}{r^2} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2 + \left\{2\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\}\right]\end{aligned} \tag{9a}$$

$$\begin{aligned}\tau_{\theta\theta} = &-p + 2\mu_{nf}\frac{u}{r} + 2\left(\alpha_1\right)_{nf}\left[\frac{u}{r}\frac{\partial u}{\partial r} + \frac{w}{r}\frac{\partial u}{\partial z} + \frac{u^2}{r^2}\right] + 4\left(\alpha_2\right)_{nf}\frac{u^2}{r^2} \\ &+ 4\left(\beta_3\right)_{nf}\frac{u}{r}\left[2\frac{u^2}{r^2} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2 + \left\{2\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right\}\right]\end{aligned} \ . \tag{9b}$$

$$\tau_{zz}=-p+2\mu_{nf}\frac{\partial w}{\partial z}+2\left(\alpha_1\right)_{nf}\left[u\frac{\partial^2 w}{\partial r\partial z}+w\frac{\partial^2 w}{\partial z^2}+2\left(\frac{\partial w}{\partial r}\right)^2+\frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\right]$$
$$+\left(\alpha_2\right)_{nf}\left[4\left(\frac{\partial w}{\partial r}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\right]+4\left(\beta_3\right)_{nf}\left(\frac{\partial w}{\partial z}\right)\begin{bmatrix}2\frac{u^2}{r^2}+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\\+\left\{2\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right\}\end{bmatrix} \tag{9c}$$

$$\tau_{rz}=\mu_{nf}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)+\left(\alpha_1\right)_{nf}\left[\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)+\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}+3\left(\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\right)\right]$$
$$+2\left(\alpha_2\right)_{nf}\left(\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)+2\left(\beta_3\right)_{nf}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\left[2\frac{u^2}{r^2}+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2+\left\{2\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right\}\right]. \tag{9d}$$

After substitution of Equations (9a-9b) into above momentum equations in Equations (7) and (8) and expansion of the resulting equations, one arrives at Equations (10) and (11)

$$\rho_{nf}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+$$
$$\left(\alpha_1\right)_{nf}\begin{bmatrix}-2\frac{u^2}{r^3}-\frac{2}{r^2}\frac{\partial u}{\partial t}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+3\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+5\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+3\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+2\frac{w}{r}\frac{\partial^2 u}{\partial r\partial z}+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^2\\-2\frac{u}{r^2}\frac{\partial u}{\partial r}+4\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+u\frac{\partial^3 u}{\partial r\partial z^2}+w\frac{\partial^3 w}{\partial r\partial z^2}+3\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}\\-2\frac{w}{r^2}\frac{\partial u}{\partial z}+\frac{\partial^3 u}{\partial z^2\partial t}+w\frac{\partial^3 u}{\partial z^3}+4\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+2w\frac{\partial^3 u}{\partial r^2\partial z}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\\+10\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial^2 u}{\partial r\partial t}+\frac{\partial^3 w}{\partial r\partial z\partial t}+u\frac{\partial^3 u}{\partial r^2\partial z}+2\frac{\partial^3 u}{\partial r^2\partial t}+2u\frac{\partial^3 u}{\partial r^3}+2\frac{u}{r}\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}\end{bmatrix}$$
$$+\left(\alpha_2\right)_{nf}\begin{bmatrix}-4\frac{u^2}{r^3}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}+4\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r}\left(\frac{\partial u}{\partial z}\right)^2+8\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}\\+2\frac{\partial^2 w}{\partial r\partial z}\frac{\partial w}{\partial z}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\end{bmatrix}$$
$$+\left(\beta_3\right)_{nf}\begin{bmatrix}-8\frac{u^3}{r^4}+8\frac{u^2}{r^2}\frac{\partial^2 u}{\partial r^2}+\frac{8}{r}\left(\frac{\partial u}{\partial r}\right)^3+24\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial r^2}+\frac{4}{r}\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+\frac{4}{r}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{8}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\\+16\frac{u}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+\frac{8}{r}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+8\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial r^2}-4\frac{u}{r^2}\left(\frac{\partial w}{\partial r}\right)^2+16\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\\+4\frac{u^2}{r^2}\frac{\partial^2 w}{\partial r\partial z}+8\frac{\partial^2 u}{\partial r^2}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z}-8\frac{u}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2}\\+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}+4\frac{u^2}{r^2}\frac{\partial^2 u}{\partial z^2}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+4\frac{u}{r^2}\left(\frac{\partial u}{\partial z}\right)^2\\+4\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial r^2}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2}-8\frac{u}{r^2}\left(\frac{\partial w}{\partial z}\right)^2+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}\\-8\frac{u^2}{r^3}\frac{\partial u}{\partial r}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+4\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial r^2}\end{bmatrix}-\frac{\sigma_{nf}B_0^2 u}{1-ct}-\frac{\mu_{nf}}{K}u \tag{10}$$

$$
\begin{aligned}
&\rho_{nf}\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial r^2}+\frac{\partial^2 w}{\partial z^2}\right)\\
&+(\alpha_1)_{nf}\left(\begin{aligned}
&\frac{u}{r}\frac{\partial^2 w}{\partial r^2}+\frac{w}{r}\frac{\partial^2 w}{\partial z^2}+\frac{\partial^3 w}{\partial r^2\partial t}+\frac{w}{r}\frac{\partial^2 w}{\partial r\partial z}+\frac{3}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+\frac{3}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\\
&+\frac{1}{r}\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}+2\frac{\partial^3 w}{\partial z^2\partial t}+w\frac{\partial^3 u}{\partial r\partial z^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r^2}+3\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}\\
&+4\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+u\frac{\partial^3 w}{\partial r^3}+\frac{u}{r}\frac{\partial^2 u}{\partial r\partial z}+u\frac{\partial^3 u}{\partial r^2\partial z}+w\frac{\partial^3 w}{\partial r^3\partial z}\\
&+3\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}+3\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2}+\frac{\partial^3 u}{\partial r\partial z\partial t}+5\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\\
&+\frac{1}{r}\frac{\partial^2 w}{\partial r\partial t}+\frac{1}{r}\frac{\partial^2 u}{\partial z\partial t}+2w\frac{\partial^3 w}{\partial z^3}+2u\frac{\partial^3 w}{\partial r\partial z^2}+8\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\\
&+4\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}.
\end{aligned}\right)\\
&+(\alpha_2)_{nf}\left(\begin{aligned}
&\frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+4\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}\\
&+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial^2 u}{\partial r^2}\frac{\partial w}{\partial r}+8\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}+\frac{2}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\\
&+\frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}
\end{aligned}\right)\\
&+(\beta_3)_{nf}\left[\begin{aligned}
&\frac{4}{r}\frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial r}\right)^2+\frac{4}{r}\frac{\partial w}{\partial r}\left(\frac{\partial u}{\partial r}\right)^2+4\frac{u^2}{r^3}\frac{\partial w}{\partial r}+8\frac{u}{r^2}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\\
&+\frac{4}{r}\frac{\partial u}{\partial z}\left(\frac{\partial w}{\partial z}\right)^2+4\frac{u^2}{r^2}\frac{\partial^2 u}{\partial r\partial z}+\frac{6}{r}\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial r}+\frac{4}{r}\frac{\partial w}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2\\
&+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial r\partial z}+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial r^2}+4\frac{u^2}{r^3}\frac{\partial u}{\partial z}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}\\
&+8\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}+\frac{6}{r}\frac{\partial u}{\partial z}\left(\frac{\partial w}{\partial r}\right)^2+8\frac{u}{r^2}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial r\partial z}\\
&+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\left(\frac{\partial u}{\partial z}\right)^3+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^3+4\frac{u^2}{r^2}\frac{\partial^2 w}{\partial r^2}+8\frac{u^2}{r^2}\frac{\partial^2 w}{\partial z^2}\\
&+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial r^2}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial r\partial z}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial r^2}-8\frac{u^2}{r^3}\frac{\partial w}{\partial r}\\
&+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+8\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial z^2}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial r\partial z}-8\frac{u^2}{r^3}\frac{\partial u}{\partial z}\\
&+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial r^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+4\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial z^2}+16\frac{u}{r^2}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\\
&+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+4\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial z^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}\\
&+24\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial z^2}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+8\frac{\partial^2 u}{\partial z^2}\frac{\partial w}{\partial z}\frac{\partial w}{\partial r}\\
&+18\frac{\partial w}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}.
\end{aligned}\right]-\frac{\mu_{nf}}{K}w
\end{aligned}
\tag{11}
$$

And the energy equation is given as

$$
\begin{aligned}
&\left(\rho C_p\right)_{nf}\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\mu\left[2\left(\frac{\partial u}{\partial r}\right)^2+2\frac{u^2}{r^2}+\left(\frac{\partial u}{\partial z}\right)^2+2\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial w}{\partial r}\right)^2\right]\\
&+\left(\alpha_1\right)_{nf}\left[\begin{array}{l}
2\frac{u^3}{r^3}+4\left(\frac{\partial u}{\partial r}\right)^3+2\frac{u}{r^2}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+w\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+u\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+u\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{uw}{r^2}\frac{\partial u}{\partial z}+u\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}\\
+\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z\partial t}+w\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+3\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+4\left(\frac{\partial w}{\partial z}\right)^3+3\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial w}{\partial z}+2u\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+w\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+3\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial z}\\
+2u\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+w\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+3\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial z}+3\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+6\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}+2\frac{u^2}{r^2}\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial t}\\
+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial t}+\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial t}+6\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2u\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}+2w\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+2w\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z\partial t}.
\end{array}\right]\\
&+\left(\alpha_2\right)_{nf}\left[4\frac{u^3}{r^3}+3\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+3\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+6\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+6\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}+4\left(\frac{\partial u}{\partial r}\right)^3+3\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial z}+4\left(\frac{\partial w}{\partial z}\right)^3+3\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial w}{\partial z}\right]\\
&+\left(\beta_3\right)_{nf}\left[\begin{array}{l}
2\left(\frac{\partial w}{\partial r}\right)^4+8\frac{u^4}{r^4}+8\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial u}{\partial z}\right)^2+2\left(\frac{\partial u}{\partial z}\right)^4+16\frac{u^2}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+8\left(\frac{\partial u}{\partial r}\right)^4+16\frac{u^2}{r^2}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+8\left(\frac{\partial u}{\partial z}\right)^3\frac{\partial w}{\partial r}+8\frac{u^2}{r^2}\left(\frac{\partial w}{\partial r}\right)^2\\
+8\frac{u^2}{r^2}\left(\frac{\partial u}{\partial z}\right)^2+8\frac{\partial u}{\partial z}\left(\frac{\partial w}{\partial r}\right)^3+12\left(\frac{\partial u}{\partial z}\right)^2\left(\frac{\partial w}{\partial r}\right)^2+16\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial w}{\partial z}\right)^2+8\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial w}{\partial r}\right)^2+8\left(\frac{\partial w}{\partial r}\right)^2\left(\frac{\partial w}{\partial z}\right)^2+16\frac{u^2}{r^2}\left(\frac{\partial w}{\partial z}\right)^2\\
+16\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2+8\left(\frac{\partial u}{\partial z}\right)^2\left(\frac{\partial w}{\partial z}\right)^2+16\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+8\left(\frac{\partial w}{\partial z}\right)^4.
\end{array}\right]\\
&+\frac{16\sigma_{nf}{}^{*}T_h{}^{3}}{3\left(k_1^{*}\right)_{nf}}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{\sigma_{nf}B_0{}^2u^2}{1-ct} \qquad . \qquad (12)
\end{aligned}
$$

The related boundary conditions are given as

$$
\begin{aligned}
&u(r,z,t)=U_w=\frac{ar}{2(1-ct)},\ w(r,z,t)=-w_0,\ \text{-}k_{nf}\frac{\partial T}{\partial z}=h_1\left(T_f-T\right)\ \text{at } z=0,\\
&u(r,z,t)=0,\ w(r,z,t)=\frac{\partial h}{\partial t}=-\frac{c}{2}\sqrt{\frac{\nu}{a(1-ct)}},\ \text{-}k_{nf}\frac{\partial T}{\partial z}=h_2\left(T-T_h\right)\ \text{at } z=h(\text{t}),
\end{aligned}
\qquad (13)
$$

where $w_0>0$ represents suction. The various physical and thermal properties in the Equation (7-10) are given as

$$\rho_{nf}=\rho_f\left(1-\phi\right)+\rho_s\phi \qquad (14)$$

$$\left(\rho c_p\right)_{nf}=\left(\rho c_p\right)_f\left(1-\phi\right)+\left(\rho c_p\right)_s\phi \qquad (15)$$

$$\left(\rho\beta\right)_{nf}=\left(\rho\beta\right)_f\left(1-\phi\right)+\left(\rho\beta\right)_s\phi \qquad (16)$$

$$\mu_{nf}=\frac{\mu_f}{(1-\phi)^{2.5}} \qquad . \qquad (17)$$

$$\sigma_{nf}=\sigma_f\left[1+\frac{3\left\{\frac{\sigma_s}{\sigma_f}-1\right\}\phi}{\left\{\frac{\sigma_s}{\sigma_f}+2\right\}\phi-\left\{\frac{\sigma_s}{\sigma_f}-1\right\}\phi}\right], \qquad (18)$$

$$k_{nf} = k_f \left[ \frac{k_s + 2k_f - 2\phi\left(k_f - k_s\right)}{k_s + 2k_f + \phi\left(k_f - k_s\right)} \right] \quad (19)$$

$$\frac{\partial q_r}{\partial y} = -\frac{4\sigma}{3K}\frac{\partial T^4}{\partial y} \cong -\frac{16\sigma T_s^3}{3K}\frac{\partial^2 T}{\partial y^2} \quad \text{(using Rosseland's approximation)} \quad (20)$$

Table 1 : Physical and thermal properties of the base fluid [66-71]

| Base fluid | $P$ (kg/m$^3$) | $c_p$ ( J/kgK) | $k$ (W/mK) |
|---|---|---|---|
| Pure water | 997.1 | 4179 | 0.613 |
| Ethylene Glycol | 1115 | 2430 | 0.253 |
| Engine oil | 884 | 1910 | 0.144 |
| Kerosene | 783 | 2010 | 0.145 |

Table 2 : Physical and thermal properties of nanoparticles [66-71]

| Nanoparticles | $P$ (kg/m$^3$) | $c_p$ ( J/kgK) | $k$ (W/mK) |
|---|---|---|---|
| Copper (Cu) | 8933 | 385 | 401 |
| Aluminum oxide ($Al_2O_3$) | 3970 | 765 | 40 |
| SWCNTs | 2600 | 42.5 | 6600 |
| Silver (Ag) | 10500 | 235.0 | 429 |
| Titanium dioxide ($TiO_2$) | 4250 | 686.2 | 8.9538 |
| Copper (II) Oxide (CuO) | 783 | 540 | 18 |

Table 1 and 2 present the physical and thermal properties of the base fluid and the nanoparticles, respectively. However, the present analysis is based on Copper (II) Oxide (CuO) and Pure water.

We consider the following transformations:

$$\eta = \frac{z}{h(t)},\ u = U_w f'(\eta),\ w = -\sqrt{\frac{av}{1-ct}}\, f(\eta),\ \theta = \frac{T - T_h}{T_f - T_h} \quad (21)$$

Identically, it is established that the continuity equation is satisfied. Omitting the pressure terms from the momentum equations in Equations (10) and (11) and using the above transformations, one obtains

$$\begin{aligned} & f^{iv} + ff''' - \frac{Sq}{2}\left(3f'' - \eta f'''\right) - \left(M^2 + \frac{1}{Da}\right) f'' + \alpha\left(-2f'f''' - f'f^{iv} - ff^{v} + \frac{Sq}{2}\left(5f^{iv} - \eta f^{v}\right)\right) \\ & -\gamma\left(2f'f''' + ff^{iv}\right) + \beta\left(7f''^3 + 24ff'f''' + 3f'^2 f^{iv} + \mathrm{Re}\left(3f'f'''^2 + \frac{3}{2}f''^2 f^{iv}\right)\right) = 0 \end{aligned} \quad (22)$$

$$(1+Rd)\theta''+\Pr\left(\theta' f-\frac{Sq}{2}\eta\theta'\right)+PrEc\left\{\begin{array}{l} M^2 f'^2+\frac{6}{\mathrm{Re}}f'^2+f''^2+\alpha\left(\begin{array}{l}\frac{1}{\mathrm{Re}}\left(-6f'^3-6fff''\right)\\ -ff''^2-fff'''+\frac{Sq}{2}\left(3f''^2+\eta ff'''\right)+\frac{Sq}{\mathrm{Re}}\left(6f'^2+3\eta ff''\right)\end{array}\right)\\ -3\gamma\left(\frac{2f'^3}{\mathrm{Re}}+\frac{ff''^2}{2}\right)+\beta\left(\frac{18}{\mathrm{Re}}f'^4+6f'^2f''^2+\frac{\mathrm{Re}}{2}f''^4\right)\end{array}\right\}=0 \tag{23}$$

The corresponding boundary conditions are

$$f(0)=A,\ f(1)=\frac{Sq}{2},\ f'(0)=1,\ f'(1)=0, \tag{24}$$

$$\theta'(0)=\Upsilon_1\left(\theta(0)-1\right),\ \theta'(1)=-\Upsilon_2\theta(1) \tag{25}$$

where

$$\begin{array}{l}\alpha=\frac{\alpha_1 a}{\mu(1-ct)},\beta=\frac{2\beta_3 a^2}{\mu(1-ct)^2},\gamma=\frac{\alpha_2 a}{\mu(1-ct)},\mathrm{Re}=\frac{ar^2}{2v(1-ct)},\\ \Pr=\frac{\mu C_p}{K},Ec=\frac{U_w^2}{C_p\left(T_f-T_h\right)},M^2=\frac{\sigma B_0^2}{\rho a},Sq=\frac{c}{a},\\ A=\sqrt{\frac{(1-ct)}{av}}w_0,Rd=\frac{16\sigma^* T_h^3}{3k_1^* K},\Upsilon_1=\frac{h_1 h(\mathrm{t})}{K},\Upsilon_2=\frac{h_2 h(\mathrm{t})}{K},\end{array} \tag{26}$$

## 3. Method of solution by homotopy perturbation method

The relative simplicity, comparative advantages, provision of acceptable analytical results with convenient convergence and stability coupled with total analytic procedures of homotopy perturbation method compel us to consider the method for solving the system of nonlinear differential equations in Equations (22) and (23).

### 3.1 The basic idea of homotopy perturbation method

In order to establish the basic idea behind homotopy perturbation method, consider a system of nonlinear differential equations given as

$$A(U)-f(r)=0,\quad r\in\Omega, \tag{27}$$

with the boundary conditions

$$B\left(u,\frac{\partial u}{\partial\eta}\right)=0,\qquad r\in\Gamma, \tag{28}$$

where $A$ is a general differential operator, $B$ is a boundary operator, $f(r)$ a known analytical function and $\Gamma$ is the boundary of the domain $\Omega$

The operator $A$ can be divided into two parts, which are $L$ and $N$, where $L$ is a linear operator, $N$ is a non-linear operator. Equation (20) can be therefore rewritten as follows

$$L(u)+N(u)-f(r)=0. \tag{29}$$

By the homotopy technique, a homotopy $U(r,p):\Omega\times[0,1]\rightarrow R$ can be constructed, which satisfies

$$H(U,p)=(1-p)\left[L(U)-L(U_o)\right]+p\left[A(U)-f(r)\right]=0, \quad p\in[0,1], \tag{30}$$

or

$$H(U,p)=L(U)-L(U_o)+pL(U_o)+p\left[N(U)-f(r)\right]=0. \tag{31}$$

In the above Equations (30) and (31), $p\in[0,1]$ is an embedding parameter, $u_o$ is an initial approximation of equation of Equation (27), which satisfies the boundary conditions.

Also, from Equation (30) and Equation (31), we will have

$$H(U,0)=L(U)-L(U_o)=0, \tag{32}$$

or

$$H(U,0)=A(U)-f(r)=0. \tag{33}$$

The changing process of $p$ from zero to unity is just that of $U(r,p)$ from $u_o(r)$ to $u(r)$. This is referred to homotopy in topology. Using the embedding parameter $p$as a small parameter, the solution of Equation (30) and Equation (31) can be assumed to be written as a power series in p as given in Equation (34)

$$U=U_o+pU_1+p^2U_2+... \tag{34}$$

It should be pointed out that of all the values of $p$ between 0 and 1, $p=1$ produces the best result. Therefore, setting $p=1$, results in the approximation solution of Equation (26)

$$u=\lim_{p\rightarrow 1}U=U_o+U_1+U_2+... \tag{35}$$

The basic idea expressed above is a combination of homotopy and perturbation method. Hence, the method is called homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantages of the traditional perturbation techniques. The series Equation (35) is convergent for most cases.

### 3.2 Application of the homotopy perturbation method to the present problem

According to homotopy perturbation method (HPM), one can construct an homotopy for Equation (22) and (23) as

$$H_1(p,\eta)=(1-p)\left[f^{iv}\right]+p\left[\begin{array}{l} f^{iv}+ff'''-\frac{Sq}{2}(3f''-\eta f''')-\left(M^2+\frac{1}{Da}\right)f''+\alpha\left(-2f''f'''-f'f^{iv}-ff^{v}+\frac{Sq}{2}\left(5f^{iv}-\eta f^{v}\right)\right) \\ -\gamma\left(2f''f'''+f'f^{iv}\right)+\beta\left(7f''^{3}+24f'f''f'''+3f'^{2}f^{iv}+\mathrm{Re}\left(3f''f'''^{2}+\frac{3}{2}f''^{2}f^{iv}\right)\right)=0 \end{array}\right]=0, \tag{36}$$

$$H_2(p,\eta)=(1-p)\left[\theta''\right]+p\left[\begin{array}{l} \theta''+\frac{Pr}{(1+Rd)}\left(\theta' f-\frac{Sq}{2}\eta\theta'\right) \\ +\frac{\Pr Ec}{(1+Rd)}\left\{\begin{array}{l} M^2 f'^{2}+\frac{6}{\mathrm{Re}}f'^{2}+f''^{2}+\alpha\left(\begin{array}{l}\frac{1}{\mathrm{Re}}\left(-6f'^{3}-6ff'f''\right)-f'f''^{2}-ff''f''' \\ +\frac{Sq}{2}\left(3f''^{2}+\eta f''f'''\right)+\frac{Sq}{\mathrm{Re}}\left(6f'^{2}+3\eta f'f''\right)\end{array}\right) \\ -3\gamma\left(\frac{2f'^{3}}{\mathrm{Re}}+\frac{f'f''^{2}}{2}\right)+\beta\left(\frac{18}{\mathrm{Re}}f'^{4}+6f'^{2}f''^{2}+\frac{\mathrm{Re}}{2}f''^{4}\right)\end{array}\right\} \end{array}\right]=0, \tag{37}$$

Taking power series of velocity, temperature and concentration fields, gives

$$f=f_0+pf_1+p^2 f_2+p^3 f_3+p^4 f_4+\dots \tag{38}$$

and

$$\theta=\theta_0+p\theta_1+p^2\theta_2+p^3\theta_3+p^4\theta_4+\dots \tag{39}$$

On substituting Equation (38) and (39) into Equations (37), we have

For the momentum equation as

$$
\begin{array}{l}
\left(1-p\right)\left\{ f_0'''' + pf_1'''' + p^2 f_2'''' + p^3 f_3'''' + p^4 f_4'''' + ...\right\} + \\
p\left\{\begin{array}{l}
(f_0'''' + pf_0'''' + p^2 f_2'''' + p^3 f_3'''' + p^4 f_4'''' + ...) + (f_0 + pf_1 + p^2 f_2 + p^3 f_3 + p^4 f_4 + ...)(f_0''' + pf_1''' + p^2 f_2''' + p^3 f_3''' + p^4 f_4''' + ...) \\
-\left(1/2\right) Sq((3 f_0'' + 3pf_1'' + 3p^2 f_2'' + 3p^3 f_3'' + 3p^4 f_4'' + ...) + \eta(f_0''' + pf_1''' + p^2 f_2''' + p^3 f_3''' + p^4 f_4''' + ...) \\
-\left(M^2 + \dfrac{1}{Da}\right)(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...) \\
-\alpha((f_0 + pf_1 + p^2 f_2 + p^3 f_3 + p^4 f_4 + ...)(f_0^{v} + pf_1^{v} + p^2 f_2^{v} + p^3 f_3^{v} + p^4 f_4^{v} + ...) \\
+2(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)(f_0''' + pf_1''' + p^2 f_2''' + p^3 f_3''' + p^4 f_4''' + ...) \\
+(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)(f_0'''' + pf_0'''' + p^2 f_2'''' + p^3 f_3'''' + p^4 f_4'''' + ...) - \\
\left(1/2\right) Sq((5 f_0^{v} + 5pf_1^{v} + 5p^2 f_2^{v} + 5p^3 f_3^{v} + 5p^4 f_4^{v} + ...) + \eta(f_0^{v} + pf_1^{v} + p^2 f_2^{v} + p^3 f_3^{v} + p^4 f_4^{v} + ...)) \\
-\gamma((2(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...))(f_0''' + pf_1''' + p^2 f_2''' + p^3 f_3''' + p^4 f_4''' + ...) \\
+(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)(f_0'''' + pf_0'''' + p^2 f_2'''' + p^3 f_3'''' + p^4 f_4'''' + ...)) + \beta(7(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)^3 \\
+(24(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...))(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)(f_0''' + pf_1''' + p^2 f_2''' + p^3 f_3''' + p^4 f_4''' + ...) \\
+3(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)^2 (f_0'''' + pf_0'''' + p^2 f_2'''' + p^3 f_3'''' + p^4 f_4'''' + ...) \\
+\mathrm{Re}(3(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)(f_0''' + pf_1''' + p^2 f_2''' + p^3 f_3''' + p^4 f_4''' + ...)^2 \\
+\left(3/2\right)(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)^2 (f_0'''' + pf_0'''' + p^2 f_2'''' + p^3 f_3'''' + p^4 f_4'''' + ...)))
\end{array}\right\} = 0
\end{array}
\tag{40}
$$

$$
\begin{array}{l}
\left(1+Rd\right)\left(1-p\right)\left\{\theta_0'' + p\theta_1'' + p^2\theta_2'' + p^3\theta_3'' + p^4\theta_4'' + ...\right\} + \\
p\left\{\begin{array}{l}
\left(1+Rd\right)(\theta_0'' + p\theta_1'' + p^2\theta_2'' + p^3\theta_3'' + p^4\theta_4'' + ...) + \\
\mathrm{Pr}((\theta_0' + p\theta_1' + p^2\theta_2' + p^3\theta_3' + p^4\theta_4' + ...)(f_0 + pf_1 + p^2 f_2 + p^3 f_3 + p^4 f_4 + ...) \\
-\left(1/2\right) Sq\eta(\theta_0' + p\theta_1' + p^2\theta_2' + p^3\theta_3' + p^4\theta_4' + ...)) \\
+PrEc(M^2(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)^2 \\
+\dfrac{6}{\mathrm{Re}}(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)^2 \\
+(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)^2 + \alpha((-6(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)^3 \\
-(6(f_0 + pf_1 + p^2 f_2 + p^3 f_3 + p^4 f_4 + ...))(f_0' + pf_1' + \\
p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)) / \mathrm{Re} \\
-(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)^2 \\
-(f_0 + pf_1 + p^2 f_2 + p^3 f_3 + p^4 f_4 + ...)(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + \\
p^4 f_4'' + ...)(f_0''' + pf_1''' + p^2 f_2''' + p^3 f_3''' + p^4 f_4''' + ...) \\
+\dfrac{Sq}{2}(3(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)^2 \\
+\eta(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)(f_0''' + pf_1''' + p^2 f_2''' + p^3 f_3''' + p^4 f_4''' + ...)) \\
+\dfrac{Sq}{\mathrm{Re}}(6(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)^2 \\
+3\eta(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)) \\
-3\gamma(\dfrac{2}{\mathrm{Re}}(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)^3 \\
+(1/2(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...))(f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)^2) \\
+\beta(\dfrac{18}{\mathrm{Re}}(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)^4 \\
+6(f_0' + pf_1' + p^2 f_2' + p^3 f_3' + p^4 f_4' + ...)^2 (f_0'' + pf_1'' + p^2 f_2'' + p^3 f_3'' + p^4 f_4'' + ...)^2 \\
+\dfrac{\mathrm{Re}}{2}(f_0'''' + pf_0'''' + p^2 f_2'''' + p^3 f_3'''' + p^4 f_4'''' + ...))))
\end{array}\right\} = 0
\end{array}
\tag{41}
$$

Grouping like terms based on the power of $p$, we have:

$$p^0 : f_0^{''''} = 0, \tag{42}$$

$$\begin{aligned} p^1 :\ & f_1^{''''} + f_0 f_0^{'''} - (1/2)Sq\left(3 f_0^{''} + \eta f_0^{'''}\right) - \left(M^2 + \frac{1}{Da}\right) f_0^{''} - \alpha\left( f_0 f_0^{v} + 2 f_0^{''} f_0^{'''} + f_0^{'} f_0^{''''} - (1/2)Sq\left(5 f_0^{v} + \eta f_0^{v}\right)\right) - \\ & \gamma\left(2 f_0^{''} f_0^{'''} + f_0^{'} f_0^{''''}\right) + \beta\left(7 f_0^{''3} + 24 f_0^{'} f_0^{''} f_0^{'''} + 9 f_0^{'2} f_0^{''''} + \mathrm{Re}\left(3 f_0^{''} f_0^{'''2} + (3/2) f_0^{''2} f_0^{''''}\right)\right) = 0, \end{aligned} \tag{43}$$

$$\begin{aligned} p^2 :\ & f_2^{''''} + f_0 f_1^{'''} + f_1 f_0^{'''} - (1/2)Sq\left(3 f_1^{''} + \eta f_1^{'''}\right) - \left(M^2 + \frac{1}{Da}\right) f_1^{''} - \alpha( f_0 f_1^{v} + f_1 f_0^{v} + 2 f_0^{''} f_1^{'''} + 2 f_1^{''} f_0^{'''} + \\ & f_0^{'} f_0^{''''} + f_1^{'} f_0^{''''} - (1/2)Sq\left(5 f_1^{v} + \eta f_1^{v}\right)) - \gamma\left(2 f_0^{''} f_1^{'''} + 2 f_1^{''} f_0^{'''} + f_0^{'} f_0^{''''} + f_1^{'} f_0^{''''}\right) + \beta(21 f_0^{''2} f_1^{''} + \\ & 24 f_0^{'} f_0^{''} f_1^{'''} + (24( f_0^{'} f_1^{''} + f_1^{'} f_0^{''})) f_0^{'''} + 9 f_0^{'2} f_0^{''''} + 18 f_0^{'} f_1^{'} f_0^{''''} + \mathrm{Re}(6 f_0^{''} f_0^{'''} f_1^{'''} + 3 f_1^{''} f_0^{'''2} + \\ & (3/2) f_0^{''2} f_0^{''''} + 3 f_0^{''} f_1^{''} f_0^{''''})) = 0, \end{aligned} \tag{44}$$

$$\begin{aligned} p^3 :\ & f_3^{''''} + f_0 f_2^{'''} + f_1 f_1^{'''} + f_2 f_0^{'''} - (1/2)Sq\left(3 f_2^{''} + \eta f_2^{'''}\right) - \left(M^2 + \frac{1}{Da}\right) f_2^{''} - \alpha( f_0 f_2^{v} + f_1 f_1^{v} + f_2 f_0^{v} + 2 f_0^{''} f_2^{'''} + \\ & 2 f_1^{''} f_1^{'''} + 2 f_2^{''} f_0^{'''} + f_0^{'} f_2^{''''} + f_1^{'} f_0^{''''} + f_2^{'} f_0^{''''} - (1/2)Sq\left(5 f_2^{v} + \eta f_2^{v}\right)) - \gamma(2 f_0^{''} f_2^{'''} + 2 f_1^{''} f_1^{'''} + 2 f_2^{''} f_0^{'''} + \\ & f_0^{'} f_2^{''''} + f_1^{'} f_0^{''''} + f_2^{'} f_0^{''''}) + \beta(7 f_0^{''}\left(2 f_0^{''} f_2^{''} + f_1^{''2}\right) + 14 f_1^{''2} f_0^{''} + 7 f_2^{''} f_0^{''2} + 24 f_0^{'} f_0^{''} f_2^{'''} + \\ & (24( f_0^{'} f_1^{''} + f_1^{'} f_0^{''})) f_1^{'''} + (24( f_0^{'} f_2^{''} + f_1^{'} f_1^{''} + f_2^{'} f_0^{''})) f_0^{'''} + 9 f_0^{'2} f_2^{''''} + 18 f_0^{'} f_1^{'} f_0^{''''} + \left(9\left(2 f_0^{'} f_2^{'} + f_1^{'2}\right)\right) f_0^{''''} + \\ & \mathrm{Re}(3 f_0^{''}\left(2 f_0^{'''} f_2^{'''} + f_1^{'''2}\right) + 6 f_1^{''} f_0^{'''} f_1^{'''} + 3 f_2^{''} f_0^{'''2} + (3/2) f_0^{''2} f_2^{''''} + 3 f_0^{''} f_1^{''} f_0^{''''} + \left(3/2\left(2 f_0^{''} f_2^{''} + f_1^{''2}\right)\right) f_0^{''''})) = 0 \end{aligned} \tag{45}$$

$$\begin{aligned} p^4 :\ & f_4^{''''} + f_0 f_3^{'''} + f_1 f_2^{'''} + f_2 f_1^{'''} + f_3 f_0^{'''} - (1/2)Sq\left(3 f_3^{''} + \eta f_3^{'''}\right) - \left(M^2 + \frac{1}{Da}\right) f_3^{''} - \alpha( f_0 f_3^{v} + f_1 f_2^{v} + f_2 f_1^{v} + f_3 f_0^{v} + 2 f_0^{''} f_3^{'''} + \\ & 2 f_1^{''} f_2^{'''} + 2 f_2^{''} f_1^{'''} + 2 f_3^{''} f_0^{'''} + f_0^{'} f_3^{''''} + f_1^{'} f_2^{''''} + f_2^{'} f_0^{''''} + f_3^{'} f_0^{''''} - \left(\frac{3}{2}\right) S_q (5 f_3^{v} + \eta f_3^{v})) - \gamma(2 f_0^{''} f_3^{'''} + 2 f_1^{''} f_2^{'''} + \\ & 2 f_2^{''} f_1^{'''} + 2 f_3^{''} f_0^{'''} + f_0^{'} f_3^{''''} + f_1^{'} f_2^{''''} + f_2^{'} f_0^{''''} + f_3^{'} f_0^{''''}) + \beta(7 f_0^{''}\left(2 f_0^{''} f_3^{''} + 2 f_1^{''} f_2^{''}\right) + 7 f_1^{''}(2 f_0^{''} f_2^{''} + f_1^{''2}) + \\ & 14 f_2^{''} f_0^{''} f_1^{''} + 7 f_3^{''} f_0^{''2} + 24 f_0^{'} f_0^{''} f_3^{'''} + \left(24\left( f_0^{'} f_1^{''} + f_1^{'} f_0^{''}\right)\right) f_2^{'''} + (24( f_0^{'} f_2^{''} + f_1^{'} f_1^{''} + f_2^{'} f_0^{''})) f_1^{'''} + (24( f_0^{'} f_3^{''} + \\ & f_1^{'} f_2^{''} + f_2^{'} f_1^{''} + f_3^{'} f_0^{''})) f_0^{'''} + 9 f_0^{'2} f_3^{''''} + 18 f_0^{'} f_1^{'} f_2^{''''} + (9(2 f_0^{'} f_2^{'} + f_1^{'2})) f_0^{''''} + \left(9\left(2 f_0^{'} f_3^{'} + 2 f_1^{'} f_2^{'}\right)\right) f_0^{''''} + \\ & \mathrm{Re}(3 f_0^{''}\left(2 f_0^{'''} f_3^{'''} + 2 f_1^{'''} f_2^{'''}\right) + 3 f_1^{''}\left(2 f_0^{'''} f_2^{'''} + f_1^{'''2}\right) + 6 f_2^{''} f_0^{'''} f_1^{'''} + 3 f_3^{''} f_0^{'''2} + \left(\frac{3}{2}\right) f_0^{''2} f_3^{''''} + 3 f_0^{''} f_1^{''} f_2^{''''} + \\ & \left(\frac{3}{2}(2 f_0^{''} f_2^{''} + f_1^{''2})) f_0^{''''} + \left(3/2\left(2 f_0^{''} f_3^{''} + 2 f_1^{''} f_2^{''}\right)\right) f_0^{''''})\right) = 0 \end{aligned} \tag{46}$$

Similarly, for the energy equation, we have

$$p^0 : (1 + Rd)\,\theta_0^{''} = 0, \tag{47}$$

$$\begin{aligned}
&p^{1}:\left(1+Rd\right)\theta_1^{\prime\prime}+\Pr\left(\theta_0^{\prime} f_0-\left(1/2\right)S\eta\theta_0^{\prime}\right)+PrEc(M^{2} f_0^{\prime 2}+6 f_0^{\prime 2}/\mathrm{Re}+f_0^{\prime\prime 2}+\alpha\left(\left(-6 f_0^{\prime 3}-6 f_0 f_0^{\prime} f_0^{\prime\prime}\right)/\mathrm{Re}-f_0^{\prime} f_0^{\prime\prime 2}-\right.\\
&\quad f_0 f_0^{\prime\prime} f_0^{\prime\prime\prime}+\left(1/2\right)S\left(3 f_0^{\prime\prime 2}+\eta f_0^{\prime\prime} f_0^{\prime\prime\prime}\right)+S\left(6 f_0^{\prime 2}+3\eta f_0^{\prime} f_0^{\prime\prime}\right)/\mathrm{Re}-3\gamma\left(2 f_0^{\prime 3}/\mathrm{Re}+\left(1/2\right) f_0^{\prime} f_0^{\prime\prime 2}\right)+\\
&\quad \beta\left(18 f_0^{\prime 4}/\mathrm{Re}+6 f_0^{\prime 2} f_0^{\prime\prime 2}+\left(1/2\right)Ref_0^{\prime\prime\prime\prime}\right)))=0
\end{aligned} \tag{48}$$

$$\begin{aligned}
&p^{2}:\left(1+Rd\right)\theta_2^{\prime\prime}+\Pr\left(\theta_0^{\prime} f_1+\theta_1^{\prime} f_0-\left(1/2\right)S\eta\theta_1^{\prime}\right)+PrEc(2M^{2} f_0^{\prime} f_1^{\prime}+12 f_0^{\prime} f_1^{\prime}/\mathrm{Re}+2 f_0^{\prime\prime} f_1^{\prime\prime}+\alpha((-18 f_0^{\prime 2} f_1^{\prime}-6 f_0 f_0^{\prime} f_1^{\prime\prime}-\\
&\quad \left(6\left(f_0 f_1^{\prime}+f_1 f_0^{\prime}\right)\right) f_0^{\prime\prime})/\mathrm{Re}-2 f_0^{\prime} f_0^{\prime\prime} f_1^{\prime\prime}-f_1^{\prime} f_0^{\prime\prime 2}-f_0 f_0^{\prime\prime} f_1^{\prime\prime\prime}-\left(f_0 f_1^{\prime\prime}+f_1 f_0^{\prime\prime}\right) f_0^{\prime\prime\prime}+\left(1/2\right)S\left(6 f_0^{\prime\prime} f_1^{\prime\prime}+\eta f_0^{\prime\prime} f_1^{\prime\prime\prime}+\eta f_1^{\prime\prime} f_0^{\prime\prime\prime}\right)+\\
&\quad S\left(12 f_0^{\prime} f_1^{\prime}+3\eta f_0^{\prime} f_1^{\prime\prime}+3\eta f_1^{\prime} f_0^{\prime\prime}\right)/\mathrm{Re}-3\gamma\left(6 f_0^{\prime 2} f_1^{\prime}/\mathrm{Re}+f_0^{\prime} f_0^{\prime\prime} f_1^{\prime\prime}+\left(1/2\right) f_1^{\prime} f_0^{\prime\prime 2}\right)+\beta(72 f_0^{\prime 3} f_1^{\prime}/\mathrm{Re}+12 f_0^{\prime 2} f_0^{\prime\prime} f_1^{\prime\prime}+\\
&\quad 12 f_0^{\prime} f_1^{\prime} f_0^{\prime\prime 2}+\left(1/2\right)Ref_1^{\prime\prime\prime\prime})))=0,
\end{aligned} \tag{49}$$

$$\begin{aligned}
&p^{3}:\left(1+Rd\right)\theta_3^{\prime\prime}+\Pr\left(\theta_0^{\prime} f_2+\theta_1^{\prime} f_1+\theta_2^{\prime} f_0-\left(1/2\right)S\eta\theta_2^{\prime}\right)+PrEc(M^{2}\left(2 f_0^{\prime} f_2^{\prime}+f_1^{\prime 2}\right)+\left(6\left(2 f_0^{\prime} f_2^{\prime}+f_1^{\prime 2}\right)\right)/\mathrm{Re}+2 f_0^{\prime\prime} f_2^{\prime\prime}+f_1^{\prime\prime 2}+\\
&\quad \alpha((-6 f_0^{\prime}\left(2 f_0^{\prime} f_2^{\prime}+f_1^{\prime 2}\right)-12 f_1^{\prime 2} f_0^{\prime}-6 f_2^{\prime} f_0^{\prime 2}-6 f_0 f_0^{\prime} f_2^{\prime\prime}-\left(6\left(f_0 f_1^{\prime}+f_1 f_0^{\prime}\right)\right) f_1^{\prime\prime}-\left(6\left(f_0 f_2^{\prime}+f_1 f_1^{\prime}+f_2 f_0^{\prime}\right)\right) f_0^{\prime\prime})/\mathrm{Re}-\\
&\quad f_0^{\prime}\left(2 f_0^{\prime\prime} f_2^{\prime\prime}+f_1^{\prime\prime 2}\right)-2 f_1^{\prime} f_0^{\prime\prime} f_1^{\prime\prime}-f_2^{\prime} f_0^{\prime\prime 2}-f_0 f_0^{\prime\prime} f_2^{\prime\prime\prime}-\left(f_0 f_1^{\prime\prime}+f_1 f_0^{\prime\prime}\right) f_1^{\prime\prime\prime}-\left(f_0 f_2^{\prime\prime}+f_1 f_1^{\prime\prime}+f_2 f_0^{\prime\prime}\right) f_0^{\prime\prime\prime}+\\
&\quad \left(1/2\right)S\left(6 f_0^{\prime\prime} f_2^{\prime\prime}+3 f_1^{\prime\prime 2}+\eta f_0^{\prime\prime} f_2^{\prime\prime\prime}+\eta f_1^{\prime\prime} f_1^{\prime\prime\prime}+\eta f_2^{\prime\prime} f_0^{\prime\prime\prime}\right)+S\left(12 f_0^{\prime} f_2^{\prime}+6 f_1^{\prime 2}+3\eta f_0^{\prime} f_2^{\prime\prime}+3\eta f_1^{\prime} f_1^{\prime\prime}+3\eta f_2^{\prime} f_0^{\prime\prime}\right)/\mathrm{Re}-\\
&\quad 3\gamma(\left(2\left(f_0^{\prime}\left(2 f_0^{\prime} f_2^{\prime}+f_1^{\prime 2}\right)+2 f_1^{\prime 2} f_0^{\prime}+f_2^{\prime} f_0^{\prime 2}\right)\right)/\mathrm{Re}+\left(1/2\right) f_0^{\prime}\left(2 f_0^{\prime\prime} f_2^{\prime\prime}+f_1^{\prime\prime 2}\right)+f_1^{\prime} f_0^{\prime\prime} f_1^{\prime\prime}+\left(1/2\right) f_2^{\prime} f_0^{\prime\prime 2})+\\
&\quad \beta(\left(18\left(2 f_0^{\prime 2}\left(2 f_0^{\prime} f_2^{\prime}+f_1^{\prime 2}\right)+4 f_0^{\prime 2} f_1^{\prime 2}\right)\right)/\mathrm{Re}+6 f_0^{\prime 2}\left(2 f_0^{\prime\prime} f_2^{\prime\prime}+f_1^{\prime\prime 2}\right)+24 f_0^{\prime} f_1^{\prime} f_0^{\prime\prime} f_1^{\prime\prime}+(6(2 f_0^{\prime} f_2^{\prime}+f_1^{\prime 2})) f_0^{\prime\prime 2}+\frac{Re}{2} f_2^{\prime\prime\prime\prime})))=0
\end{aligned} \tag{50}$$

$$\begin{aligned}
&p^{4}:\left(1+Rd\right)\theta_4^{\prime\prime}+\Pr\left(\theta_0^{\prime} f_3+\theta_1^{\prime} f_2+\theta_2^{\prime} f_1+\theta_3^{\prime} f_0-\left(1/2\right)S\eta\theta_3^{\prime}\right)+PrEc(M^{2}\left(2 f_0^{\prime} f_3^{\prime}+2 f_1^{\prime} f_2^{\prime}\right)+\left(6\left(2 f_0^{\prime} f_3^{\prime}+2 f_1^{\prime} f_2^{\prime}\right)\right)/\mathrm{Re}+2 f_0^{\prime\prime} f_3^{\prime\prime}+\\
&\quad 2 f_1^{\prime\prime} f_2^{\prime\prime}+\alpha((-6 f_0^{\prime}\left(2 f_0^{\prime} f_3^{\prime}+2 f_1^{\prime} f_2^{\prime}\right)-6 f_1^{\prime}\left(2 f_0^{\prime} f_2^{\prime}+f_1^{\prime 2}\right)-12 f_2^{\prime} f_0^{\prime} f_1^{\prime}-6 f_3^{\prime} f_0^{\prime 2}-6 f_0 f_0^{\prime} f_3^{\prime\prime}-\left(6\left(f_0 f_1^{\prime}+f_1 f_0^{\prime}\right)\right) f_2^{\prime\prime}-\\
&\quad \left(6\left(f_0 f_2^{\prime}+f_1 f_1^{\prime}+f_2 f_0^{\prime}\right)\right) f_1^{\prime\prime}-(6(f_0 f_3^{\prime}+f_1 f_2^{\prime}+f_2 f_1^{\prime}+f_3 f_0^{\prime})) f_0^{\prime\prime})/\mathrm{Re}-f_0^{\prime}\left(2 f_0^{\prime\prime} f_3^{\prime\prime}+2 f_1^{\prime\prime} f_2^{\prime\prime}\right)-f_1^{\prime}\left(2 f_0^{\prime\prime} f_2^{\prime\prime}+f_1^{\prime\prime 2}\right)-\\
&\quad 2 f_2^{\prime} f_0^{\prime\prime} f_1^{\prime\prime}-f_3^{\prime} f_0^{\prime\prime 2}-f_0 f_0^{\prime\prime} f_3^{\prime\prime\prime}-\left(f_0 f_1^{\prime\prime}+f_1 f_0^{\prime\prime}\right) f_2^{\prime\prime\prime}-\left(f_0 f_2^{\prime\prime}+f_1 f_1^{\prime\prime}+f_2 f_0^{\prime\prime}\right) f_1^{\prime\prime\prime}-\left(f_0 f_3^{\prime\prime}+f_1 f_2^{\prime\prime}+f_2 f_1^{\prime\prime}+f_3 f_0^{\prime\prime}\right) f_0^{\prime\prime\prime}+\\
&\quad \left(1/2\right)Sq\left(6 f_0^{\prime\prime} f_3^{\prime\prime}+6 f_1^{\prime\prime} f_2^{\prime\prime}+\eta f_0^{\prime\prime} f_3^{\prime\prime\prime}+\eta f_1^{\prime\prime} f_2^{\prime\prime\prime}+\eta f_2^{\prime\prime} f_1^{\prime\prime\prime}+\eta f_3^{\prime\prime} f_0^{\prime\prime\prime}\right)+\frac{Sq}{\mathrm{Re}}\begin{pmatrix} 12 f_0^{\prime} f_3^{\prime}+12 f_1^{\prime} f_2^{\prime}+3\eta f_0^{\prime} f_3^{\prime\prime}+ \\ 3\eta f_1^{\prime} f_2^{\prime\prime}+3\eta f_2^{\prime} f_1^{\prime\prime}+3\eta f_3^{\prime} f_0^{\prime\prime} \end{pmatrix}-\\
&\quad 3\gamma((2(f_0^{\prime}\left(2 f_0^{\prime} f_3^{\prime}+2 f_1^{\prime} f_2^{\prime}\right)+f_1^{\prime}\left(2 f_0^{\prime} f_2^{\prime}+f_1^{\prime 2}\right)+2 f_2^{\prime} f_0^{\prime} f_1^{\prime}+f_3^{\prime} f_0^{\prime 2}))/\mathrm{Re}+\left(1/2\right) f_0^{\prime}\left(2 f_0^{\prime\prime} f_3^{\prime\prime}+2 f_1^{\prime\prime} f_2^{\prime\prime}\right)+\\
&\quad \left(1/2\right) f_1^{\prime}\left(2 f_0^{\prime\prime} f_2^{\prime\prime}+f_1^{\prime\prime 2}\right)+f_2^{\prime} f_0^{\prime\prime} f_1^{\prime\prime}+\left(1/2\right) f_3^{\prime} f_0^{\prime\prime 2})+\beta(\left(18\left(2 f_0^{\prime 2}\left(2 f_0^{\prime} f_3^{\prime}+2 f_1^{\prime} f_2^{\prime}\right)+4 f_0^{\prime} f_1^{\prime}\left(2 f_0^{\prime} f_2^{\prime}+f_1^{\prime 2}\right)\right)\right)/\mathrm{Re}+\\
&\quad 6 f_0^{\prime 2}\left(2 f_0^{\prime\prime} f_3^{\prime\prime}+2 f_1^{\prime\prime} f_2^{\prime\prime}\right)+12 f_0^{\prime} f_1^{\prime}\left(2 f_0^{\prime\prime} f_2^{\prime\prime}+f_1^{\prime\prime 2}\right)+\left(12\left(2 f_0^{\prime} f_2^{\prime}+f_1^{\prime 2}\right)\right) f_0^{\prime\prime} f_1^{\prime\prime}+\left(6\left(2 f_0^{\prime} f_3^{\prime}+2 f_1^{\prime} f_2^{\prime}\right)\right) f_0^{\prime\prime 2}+\frac{Re}{2} f_3^{\prime\prime\prime\prime})))=0,
\end{aligned} \tag{51}$$

The corresponding boundary conditions are

$$f_0(0)=A,\ f_0(1)=\frac{Sq}{2},\ f_0^{\prime}(0)=1,\ f_0^{\prime}(1)=0, \tag{52a}$$

$$\theta_0^{\prime}(0)=\Upsilon_1\left(\theta_0(0)-1\right),\ \theta_0^{\prime}(1)=-\Upsilon_2\theta_0(1) \tag{52b}$$

$$f_1(0)=0,\ f_1(1)=0,\ f_1'(0)=0,\ f_1'(1)=0, \tag{53a}$$

$$\theta_1'(0)=\Upsilon_1\theta_1(0),\ \ \theta_1'(1)=-\Upsilon_2\theta_1(1) \tag{53b}$$

$$f_2(0)=0,\ f_2(1)=0,\ f_2'(0)=0,\ f_2'(1)=0, \tag{54a}$$

$$\theta_2'(0)=\Upsilon_1\theta_2(0),\ \ \theta_1'(1)=-\Upsilon_2\theta_2(1) \tag{54b}$$

$$f_3(0)=0,\ f_3(1)=0,\ f_3'(0)=0,\ f_3'(1)=0, \tag{55a}$$

$$\theta_3'(0)=\Upsilon_1\theta_3(0),\ \ \theta_3'(1)=-\Upsilon_2\theta_3(1) \tag{55b}$$

$$f_4(0)=0,\ f_4(1)=0,\ f_4'(0)=0,\ f_4'(1)=0, \tag{56a}$$

$$\theta_4'(0)=\Upsilon_1\theta_4(0),\ \ \theta_4'(1)=-\Upsilon_2\theta_4(1) \tag{56b}$$

Solving Equation (42) and applying the boundary condition Equation (52a), gives

$$f_0(\eta)=A+\eta+\left(\frac{3Sq}{2}-3A-2\right)\eta^2+\left(1+2A-Sq\right)\eta^3 \tag{57}$$

Also, when Equation (43) is solved using the boundary conditions in Equation (53a), we arrived at Equation (58)

$$f_1=\frac{1}{2}\left(\begin{aligned}&-\frac{2}{35}-\frac{3A}{7}-\frac{22A^2}{35}-\frac{A\left(M^2+\frac{1}{Da}\right)}{10}-\frac{504AReSq\beta}{5}+\frac{342ARe\beta}{5}+\frac{5ASq}{28}+\frac{12ASq\alpha}{5}+\frac{288ASq\beta}{35}+\frac{12ASq\gamma}{5}\\&-\frac{22A\alpha}{5}+\frac{654A\beta}{35}-\frac{22A\gamma}{5}-\frac{2\left(M^2+\frac{1}{Da}\right)}{15}-\frac{171ReSq\beta}{5}-\frac{324A^2ReSq\beta}{5}+\frac{72Re\beta}{5}+\frac{504A^2Re\beta}{5}+\frac{216A^3Re\beta}{5}\\&-\frac{19Sq}{140}+\frac{\left(M^2+\frac{1}{Da}\right)Sq}{20}+\frac{19Sq^2}{280}+\frac{11Sq\alpha}{5}-\frac{327Sq\beta}{35}+\frac{729A^2Sq\beta}{35}+\frac{11Sq\gamma}{5}-\frac{8\alpha}{5}-\frac{12A^2\alpha}{5}-\frac{3Sq^2\alpha}{5}+\frac{1076\beta}{105}-\frac{288A^2\beta}{35}-\\&+\frac{486A^3\beta}{35}+\frac{162AReSq^2\beta}{5}-\frac{729ASq^2\beta}{70}+\frac{126ReSq^2\beta}{5}-\frac{27ReSq^3\beta}{5}-\frac{72Sq^2\beta}{35}+\frac{243Sq^3\beta}{140}-\frac{8\gamma}{5}-\frac{12A^2\gamma}{5}-\frac{3Sq^2\gamma}{5}\end{aligned}\right)\eta^2+$$

$$\frac{1}{6}\left(\begin{aligned}&\frac{11}{35}+\frac{20A}{7}+\frac{156A^2}{35}+\frac{6A\left(M^2+\frac{1}{Da}\right)}{5}+\frac{4968AReSq\beta}{5}-\frac{3024ARe\beta}{5}-\frac{39ASq}{70}-\frac{144ASq\alpha}{5}+\frac{7866ASq\beta}{35}-\frac{144ASq\gamma}{5}+\frac{204A\alpha}{5}-\frac{13572A\beta}{35}+\\&\frac{204A\gamma}{5}+\frac{11\left(M^2+\frac{1}{Da}\right)}{10}+\frac{1512ReSq\beta}{5}+\frac{3888A^2ReSq\beta}{5}-\frac{594Re\beta}{5}-\frac{4968A^2Re\beta}{5}-\frac{2592A^3Re\beta}{5}+\frac{89Sq}{70}-\frac{3\left(M^2+\frac{1}{Da}\right)Sq}{5}-\frac{117Sq^2}{140}-\\&\frac{102Sq\alpha}{5}+\frac{6786Sq\beta}{35}-\frac{2592A^2Sq\beta}{35}-\frac{102Sq\gamma}{5}+\frac{66\alpha}{5}+\frac{144A^2\alpha}{5}+\frac{36Sq^2\alpha}{5}-\frac{4726\beta}{35}-\frac{7866A^2\beta}{35}+\frac{1728A^3\beta}{35}-\frac{1944AReSq^2\beta}{5}+\frac{1296ASq^2\beta}{35}\\&-\frac{1242ReSq^2\beta}{5}+\frac{324ReSq^3\beta}{5}-\frac{3933Sq^2\beta}{70}-\frac{216Sq^3\beta}{35}+\frac{66\gamma}{5}+\frac{144A^2\gamma}{5}+\frac{36Sq^2\gamma}{5}\end{aligned}\right)\eta^3+$$

$$\left(\begin{aligned}&180AReSq\beta\eta^4+\frac{324AReSq\beta\eta^5}{5}+99ARe\beta\eta^4-\frac{162ARe\beta\eta^5}{5}+6ASq\alpha\eta^4-\frac{12ASq\alpha\eta^5}{5}-198ASq\beta\eta^4+\frac{1629ASq\beta\eta^5}{5}-228ASq\beta\eta^6+\\&\frac{2052ASq\beta\eta^7}{35}+6ASq\gamma\eta^4+\frac{12ASq\gamma\eta^5}{5}-7A\alpha\eta^4+\frac{12A\alpha\eta^5}{5}+168A\beta\eta^4-\frac{1044A\beta\eta^5}{5}+\frac{627A\beta\eta^6}{5}-\frac{1026A\beta\eta^7}{35}-7A\gamma\eta^4+\frac{12A\gamma\eta^5}{5}-\\&\frac{99ReSq\beta\eta^4}{2}-162A^2ReSq\beta\eta^4+\frac{81ReSq\beta\eta^5}{5}+\frac{324A^2ReSq\beta\eta^5}{5}+18Re\beta\eta^4+180A^2Re\beta\eta^4+108A^3Re\beta\eta^4-\frac{27Re\beta\eta^5}{5}-\frac{324A^2Re\beta\eta^5}{5}-\\&\frac{216A^3Re\beta\eta^5}{5}+\frac{7Sq\alpha\eta^4}{2}-\frac{6Sq\alpha\eta^5}{5}-84Sq\beta\eta^4-\frac{189A^2Sq\beta\eta^4}{2}+\frac{522Sq\beta\eta^5}{5}+243A^2Sq\beta\eta^5-\frac{627Sq\beta\eta^6}{10}-\frac{1026A^2Sq\beta\eta^6}{5}+\frac{513Sq\beta\eta^7}{35}+\\&\frac{2052A^2Sq\beta\eta^7}{35}+\frac{7Sq\gamma\eta^4}{2}-\frac{6Sq\gamma\eta^5}{5}+\frac{\left(M^2+\frac{1}{Da}\right)Sq\eta^4}{8}-\frac{\left(M^2+\frac{1}{Da}\right)Sq\eta^5}{20}-2\alpha\eta^4-6A^2\alpha\eta^4-\frac{3Sq^2\alpha\eta^4}{2}+\frac{3\alpha\eta^5}{5}-162A^3\beta\eta^5+\frac{12A^2\alpha\eta^5}{5}+\\&\frac{3Sq^2\alpha\eta^5}{5}+\frac{128\beta\eta^4}{3}+198A^2\beta\eta^4+63A^3\beta\eta^4+81AReSq^2\beta\eta^4+\frac{189ASq^2\beta\eta^4}{4}+45ReSq^2\beta\eta^4-\frac{27ReSq^3\beta\eta^4}{2}+\frac{99Sq^2\beta\eta^4}{2}-\frac{63Sq^3\beta\eta^4}{8}-\\&\frac{216\beta\eta^5}{5}-\frac{1629A^2\beta\eta^5}{5}-\frac{162AReSq^2\beta\eta^5}{5}-\frac{243ASq^2\beta\eta^5}{2}-\frac{81ReSq^2\beta\eta^5}{5}+\frac{27ReSq^3\beta\eta^5}{5}-\frac{1629Sq^2\beta\eta^5}{20}+\frac{81Sq^3\beta\eta^5}{4}+\frac{114\beta\eta^6}{5}+\\&228A^2\beta\eta^6+\frac{684A^3\beta\eta^6}{5}+\frac{513ASq^2\beta\eta^6}{5}+57Sq^2\beta\eta^6-\frac{171Sq^3\beta\eta^6}{10}-\frac{171\beta\eta^7}{35}-\frac{2052A^2\beta\eta^7}{35}-\frac{1368A^3\beta\eta^7}{35}-\frac{1026ASq^2\beta\eta^7}{35}-\frac{513Sq^2\beta\eta^7}{35}+\\&\frac{171Sq^3\beta\eta^7}{35}-2\gamma\eta^4-6A^2\gamma\eta^4-\frac{3Sq^2\gamma\eta^4}{2}+\frac{3\gamma\eta^5}{5}+\frac{12A^2\gamma\eta^5}{5}+\frac{3Sq^2\gamma\eta^5}{5}-\frac{A\eta^4}{4}-\frac{A^2\eta^4}{2}-\frac{A\left(M^2+\frac{1}{Da}\right)\eta^4}{4}-\frac{ASq\eta^4}{8}-\frac{\left(M^2+\frac{1}{Da}\right)\eta^4}{6}-\frac{Sq\eta^4}{4}+\\&\frac{3Sq^2\eta^4}{16}-\frac{\eta^5}{20}-\frac{A\eta^5}{10}+\frac{A\left(M^2+\frac{1}{Da}\right)\eta^5}{10}+\frac{ASq\eta^5}{5}+\frac{\left(M^2+\frac{1}{Da}\right)\eta^5}{20}+\frac{3Sq\eta^5}{20}-\frac{Sq^2\eta^5}{10}+\frac{\eta^6}{30}+\frac{7A\eta^6}{60}+\frac{A^2\eta^6}{10}-\frac{ASq\eta^6}{10}-\frac{7Sq\eta^6}{120}+\frac{Sq^2\eta^6}{40}-\frac{\eta^7}{140}-\\&\frac{A\eta^7}{35}-\frac{A^2\eta^7}{35}+\frac{ASq\eta^7}{35}+\frac{Sq\eta^7}{70}-\frac{Sq^2\eta^7}{140}\end{aligned}\right)$$

(58)

In the same manner, the Equations (44), (45) and (46) are solved using the corresponding boundary conditions in Equations (54a), (55a) and (56a). However, the solutions are too big to be included in this paper.

Using the definition in Equation (35), the solution of the Equation (22) is given as

$$f\left(\eta\right)=\begin{pmatrix} A+\eta+\left(\dfrac{3Sq}{2}-3A-2\right)\eta^{2} \\ +\left(1+2A-Sq\right)\eta^{3}\end{pmatrix}+\frac{1}{2}\begin{pmatrix} -\dfrac{2}{35}-\dfrac{3A}{7}-\dfrac{22A^{2}}{35}-\dfrac{A\left(M^{2}+\dfrac{1}{Da}\right)}{10}-\dfrac{504AReSq\beta}{5}+\dfrac{342ARe\beta}{5}+\dfrac{5ASq}{28}+\dfrac{12ASq\alpha}{5}+\dfrac{288ASq\beta}{35}+\dfrac{12ASq\gamma}{5} \\ -\dfrac{22A\alpha}{5}+\dfrac{654A\beta}{35}-\dfrac{22A\gamma}{5}-\dfrac{2\left(M^{2}+\dfrac{1}{Da}\right)}{15}-\dfrac{171ReSq\beta}{5}-\dfrac{324A^{2}ReSq\beta}{5}+\dfrac{72Re\beta}{5}+\dfrac{504A^{2}Re\beta}{5}+\dfrac{216A^{3}Re\beta}{5} \\ -\dfrac{19Sq}{140}+\dfrac{\left(M^{2}+\dfrac{1}{Da}\right)Sq}{20}+\dfrac{19Sq^{2}}{280}+\dfrac{11Sq\alpha}{5}-\dfrac{327Sq\beta}{35}+\dfrac{729A^{2}Sq\beta}{35}+\dfrac{11Sq\gamma}{5}-\dfrac{8\alpha}{5}-\dfrac{12A^{2}\alpha}{5}-\dfrac{3Sq^{2}\alpha}{5}+\dfrac{1076\beta}{105}-\dfrac{288A^{2}\beta}{35}- \\ +\dfrac{486A^{3}\beta}{35}+\dfrac{162AReSq^{2}\beta}{5}-\dfrac{729ASq^{2}\beta}{70}+\dfrac{126ReSq^{2}\beta}{5}-\dfrac{27ReSq^{3}\beta}{5}-\dfrac{72Sq^{2}\beta}{35}+\dfrac{243Sq^{3}\beta}{140}-\dfrac{8\gamma}{5}-\dfrac{12A^{2}\gamma}{5}-\dfrac{3Sq^{2}\gamma}{5}\end{pmatrix}\eta^{2}+$$

$$\frac{1}{6}\begin{pmatrix} \dfrac{11}{35}+\dfrac{20A}{7}+\dfrac{156A^{2}}{35}+\dfrac{6A\left(M^{2}+\dfrac{1}{Da}\right)}{5}+\dfrac{4968AReSq\beta}{5}-\dfrac{3024ARe\beta}{5}-\dfrac{39ASq}{70}-\dfrac{144ASq\alpha}{5}+\dfrac{7866ASq\beta}{35}-\dfrac{144ASq\gamma}{5}+\dfrac{204A\alpha}{5}-\dfrac{13572A\beta}{35}+ \\ \dfrac{204A\gamma}{5}+\dfrac{11\left(M^{2}+\dfrac{1}{Da}\right)}{10}+\dfrac{1512ReSq\beta}{5}+\dfrac{3888A^{2}ReSq\beta}{5}-\dfrac{594Re\beta}{5}-\dfrac{4968A^{2}Re\beta}{5}-\dfrac{2592A^{3}Re\beta}{5}+\dfrac{89Sq}{70}-\dfrac{3\left(M^{2}+\dfrac{1}{Da}\right)Sq}{5}-\dfrac{117Sq^{2}}{140}- \\ \dfrac{102Sq\alpha}{5}+\dfrac{6786Sq\beta}{35}-\dfrac{2592A^{2}Sq\beta}{35}-\dfrac{102Sq\gamma}{5}+\dfrac{66\alpha}{5}+\dfrac{144A^{2}\alpha}{5}+\dfrac{36Sq^{2}\alpha}{5}-\dfrac{4726\beta}{35}-\dfrac{7866A^{2}\beta}{35}+\dfrac{1728A^{3}\beta}{35}-\dfrac{1944AReSq^{2}\beta}{5}+\dfrac{1296ASq^{2}\beta}{35} \\ -\dfrac{1242ReSq^{2}\beta}{5}+\dfrac{324ReSq^{3}\beta}{5}-\dfrac{3933Sq^{2}\beta}{70}-\dfrac{216Sq^{3}\beta}{35}+\dfrac{66\gamma}{5}+\dfrac{144A^{2}\gamma}{5}+\dfrac{36Sq^{2}\gamma}{5}\end{pmatrix}\eta^{3}+$$

$$\begin{pmatrix} 180AReSq\beta\eta^{4}+\dfrac{324AReSq\beta\eta^{5}}{5}+99ARe\beta\eta^{4}-\dfrac{162ARe\beta\eta^{5}}{5}+6ASq\alpha\eta^{4}-\dfrac{12ASq\alpha\eta^{5}}{5}-198ASq\beta\eta^{4}+\dfrac{1629ASq\beta\eta^{5}}{5}-228ASq\beta\eta^{6}+ \\ \dfrac{2052ASq\beta\eta^{7}}{35}+6ASq\gamma\eta^{4}+\dfrac{12ASq\gamma\eta^{5}}{5}-7A\alpha\eta^{4}+\dfrac{12A\alpha\eta^{5}}{5}+168A\beta\eta^{4}-\dfrac{1044A\beta\eta^{5}}{5}+\dfrac{627A\beta\eta^{6}}{5}-\dfrac{1026A\beta\eta^{7}}{35}-7A\gamma\eta^{4}+\dfrac{12A\gamma\eta^{5}}{5}- \\ \dfrac{99ReSq\beta\eta^{4}}{2}-162A^{2}ReSq\beta\eta^{4}+\dfrac{81ReSq\beta\eta^{5}}{5}+\dfrac{324A^{2}ReSq\beta\eta^{5}}{5}+18Re\beta\eta^{4}+180A^{2}Re\beta\eta^{4}+108A^{3}Re\beta\eta^{4}-\dfrac{27Re\beta\eta^{5}}{5}-\dfrac{324A^{2}Re\beta\eta^{5}}{5}- \\ \dfrac{216A^{3}Re\beta\eta^{5}}{5}+\dfrac{7Sq\alpha\eta^{4}}{2}-\dfrac{6Sq\alpha\eta^{5}}{5}-84Sq\beta\eta^{4}-\dfrac{189A^{2}Sq\beta\eta^{4}}{2}+\dfrac{522Sq\beta\eta^{5}}{5}+243A^{2}Sq\beta\eta^{5}-\dfrac{627Sq\beta\eta^{6}}{10}-\dfrac{1026A^{2}Sq\beta\eta^{6}}{5}+\dfrac{513Sq\beta\eta^{7}}{35}+ \\ \dfrac{2052A^{2}Sq\beta\eta^{7}}{35}+\dfrac{7Sq\gamma\eta^{4}}{2}-\dfrac{6Sq\gamma\eta^{5}}{5}+\dfrac{\left(M^{2}+\dfrac{1}{Da}\right)Sq\eta^{4}}{8}-\dfrac{\left(M^{2}+\dfrac{1}{Da}\right)Sq\eta^{5}}{20}-2\alpha\eta^{4}-6A^{2}\alpha\eta^{4}-\dfrac{3Sq^{2}\alpha\eta^{4}}{2}+\dfrac{3\alpha\eta^{5}}{5}-162A^{3}\beta\eta^{5}+\dfrac{12A^{2}\alpha\eta^{5}}{5}+ \\ \dfrac{3Sq^{2}\alpha\eta^{5}}{5}+\dfrac{128\beta\eta^{4}}{3}+198A^{2}\beta\eta^{4}+63A^{3}\beta\eta^{4}+81AReSq^{2}\beta\eta^{4}+\dfrac{189ASq^{2}\beta\eta^{4}}{4}+45ReSq^{2}\beta\eta^{4}-\dfrac{27ReSq^{3}\beta\eta^{4}}{2}+\dfrac{99Sq^{2}\beta\eta^{4}}{2}-\dfrac{63Sq^{3}\beta\eta^{4}}{8}- \\ \dfrac{216\beta\eta^{5}}{5}-\dfrac{1629A^{2}\beta\eta^{5}}{5}-\dfrac{162AReSq^{2}\beta\eta^{5}}{5}-\dfrac{243ASq^{2}\beta\eta^{5}}{2}-\dfrac{81ReSq^{2}\beta\eta^{5}}{5}+\dfrac{27ReSq^{3}\beta\eta^{5}}{5}-\dfrac{1629Sq^{2}\beta\eta^{5}}{20}+\dfrac{81Sq^{3}\beta\eta^{5}}{4}+\dfrac{114\beta\eta^{6}}{5}+ \\ 228A^{2}\beta\eta^{6}+\dfrac{684A^{3}\beta\eta^{6}}{5}+\dfrac{513ASq^{2}\beta\eta^{6}}{5}+57Sq^{2}\beta\eta^{6}-\dfrac{171Sq^{3}\beta\eta^{6}}{10}-\dfrac{171\beta\eta^{7}}{35}-\dfrac{2052A^{2}\beta\eta^{7}}{35}-\dfrac{1368A^{3}\beta\eta^{7}}{35}-\dfrac{1026ASq^{2}\beta\eta^{7}}{35}-\dfrac{513Sq^{2}\beta\eta^{7}}{35}+ \\ \dfrac{171Sq^{3}\beta\eta^{7}}{35}-2\gamma\eta^{4}-6A^{2}\gamma\eta^{4}-\dfrac{3Sq^{2}\gamma\eta^{4}}{2}+\dfrac{3\gamma\eta^{5}}{5}+\dfrac{12A^{2}\gamma\eta^{5}}{5}+\dfrac{3Sq^{2}\gamma\eta^{5}}{5}-\dfrac{A\eta^{4}}{4}-\dfrac{A^{2}\eta^{4}}{2}-\dfrac{A\left(M^{2}+\dfrac{1}{Da}\right)\eta^{4}}{4}-\dfrac{ASq\eta^{4}}{8}-\dfrac{\left(M^{2}+\dfrac{1}{Da}\right)\eta^{4}}{6}-\dfrac{Sq\eta^{4}}{4}+ \\ \dfrac{3Sq^{2}\eta^{4}}{16}-\dfrac{\eta^{5}}{20}-\dfrac{A\eta^{5}}{10}+\dfrac{A\left(M^{2}+\dfrac{1}{Da}\right)\eta^{5}}{10}+\dfrac{ASq\eta^{5}}{5}+\dfrac{\left(M^{2}+\dfrac{1}{Da}\right)\eta^{5}}{20}+\dfrac{3Sq\eta^{5}}{20}-\dfrac{Sq^{2}\eta^{5}}{10}+\dfrac{\eta^{6}}{30}+\dfrac{7A\eta^{6}}{60}+\dfrac{A^{2}\eta^{6}}{10}-\dfrac{ASq\eta^{6}}{10}-\dfrac{7Sq\eta^{6}}{120}+\dfrac{Sq^{2}\eta^{6}}{40}-\dfrac{\eta^{7}}{140}- \\ \dfrac{A\eta^{7}}{35}-\dfrac{A^{2}\eta^{7}}{35}+\dfrac{ASq\eta^{7}}{35}+\dfrac{Sq\eta^{7}}{70}-\dfrac{Sq^{2}\eta^{7}}{140}\end{pmatrix}$$

(59)

Now, on solving Equation (47) applying the boundary condition Equation (52b), yields

$$\theta_{0}\left(\eta\right)=\frac{\gamma_{1}\left(1+\gamma_{2}\right)-\gamma_{1}\gamma_{2}\eta}{\gamma_{1}+\gamma_{1}\gamma_{2}+\gamma_{2}} \tag{60}$$

Also, when Equation (48) is solved using the boundary conditions in Equation (53b), we derived the solution for the first-order equation for the temperature distribution. However, as a result of the large expression involved in the solution, we represent the first-order solution as

$$\theta_{1}=\theta_{1}^{a1}+\theta_{1}^{a2}+\theta_{1}^{b}+\theta_{1}^{c1}+\theta_{1}^{c2}+\theta_{1}^{d} \tag{61}$$

where

$$\theta_1^{a1} = \left( \begin{array}{l}
-\frac{1}{1+Rd}(PrEc(\frac{(-4-6A+3Sq+(6+12A-6Sq)\eta)^4}{12(6+12A-6Sq)^2}+2M^2(\frac{A\eta^2}{2}+\frac{1}{6}(1+A\begin{pmatrix}-4-6A+\\3Sq\end{pmatrix}))\eta^3+\frac{1}{12}(-6-9A+A\begin{pmatrix}3+6A-\\3Sq\end{pmatrix}+\frac{9Sq}{2})\eta^4+\frac{1}{20}(4+8A-4Sq+ \\
(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq))\eta^5+\frac{1}{30}((3+6A-3Sq)(-2-3A+\frac{3Sq}{2})+(1+2A-Sq)(-4-6A+3Sq))\eta^6+\frac{1}{42}(3+6A-3Sq)(1+2A-Sq)\eta^7)+ \\
\frac{12}{\mathrm{Re}}(\frac{A\eta^2}{2}+\frac{1}{6}(1+A(-4-6A+3Sq)))\eta^3+\frac{1}{12}(-6-9A+A(3+6A-3Sq)+\frac{9Sq}{2})\eta^4+\frac{1}{20}(4+8A-4Sq+(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq))\eta^5+ \\
\frac{1}{30}((3+6A-3Sq)(-2-3A+\frac{3Sq}{2})+(1+2A-Sq)(-4-6A+3Sq))\eta^6+\frac{1}{42}(3+6A-3Sq)(1+2A-Sq)\eta^7)+\alpha(-\eta^2-A(-4-6A+3Sq)\eta^2- \\
\frac{1}{6}(-24-36A+2A(6+12A-6Sq)+18Sq+(-4-6A+3Sq)(2+2A(-4-6A+3Sq)))\eta^3-\frac{1}{12}(24+48A-24Sq+2(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq)+ \\
2(-4-6A+3Sq)^2+(-4-6A+3Sq)\begin{pmatrix}-24-36A+\\2A(6+12A-6Sq)+18Sq\end{pmatrix}+(3+6A-3Sq)(2+2A(-4-6A+3Sq)))\eta^4-\frac{1}{20}(2(6+12A-6Sq)(-2-3A+\frac{3Sq}{2})+ \\
4(3+6A-3Sq)(-4-6A+3Sq)+2(1+2A-Sq)(-4-6A+3Sq)+(3+6A-3Sq)(-24-36A+2A(6+12A-6Sq)+18Sq)+(-4-6A+3Sq) \\
(24+48A-24Sq+2(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq)+2(-4-6A+3Sq)^2))\eta^5-\frac{1}{30}(2(3+6A-3Sq)^2+2(6+12A-6Sq)(1+2A-Sq)+(-4-6A+3Sq) \\
(2(6+12A-6Sq)(-2-3A+\frac{3Sq}{2})+4(3+6A-3Sq)(-4-6A+3Sq)+2(1+2A-Sq)(-4-6A+3Sq))+(3+6A-3Sq)(24+48A-24Sq+ \\
2(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq)+2(-4-6A+3Sq)^2))\eta^6-\frac{1}{42}((2(3+6A-3Sq)^2+2(6+12A-6Sq)(1+2A-Sq))(-4-6A+3Sq)+ \\
(3+6A-3Sq)(2(6+12A-6Sq)(-2-3A+\frac{3Sq}{2})+4(3+6A-3Sq)(-4-6A+3Sq)+2(1+2A-Sq)(-4-6A+3Sq)))\eta^7- \\
\frac{1}{56}(2(3+6A-3Sq)^2+2(6+12A-6Sq)(1+2A-Sq))(3+6A-3Sq)\eta^8-6+12A-6Sq)(\frac{1}{2}A(-4-6A+3Sq)\eta^2+\frac{1}{6}(-4-6A+A(6+12A-6Sq)+3Sq)\eta^3+ \\
\frac{1}{12}(6+12A-6Sq+(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq))\eta^4+\frac{1}{20}((6+12A-6Sq)(-2-3A+\frac{3Sq}{2})+(1+2A-Sq)(-4-6A+3Sq))\eta^5+ \\
\frac{1}{30}(6+12A-6Sq)(1+2A-Sq)\eta^6)+\frac{1}{\mathrm{Re}}Sq(6A\eta^2+\frac{1}{2}(-4-6A+3Sq)\eta^3+2(1+A(-4-6A+3Sq))\eta^3+(-6-9A+A(3+6A-3Sq)+\frac{9Sq}{2})\eta^4+ \\
\frac{1}{4}(6+12A-6Sq+(-4-6A+3Sq)^2)\eta^4+\frac{3}{20}((6+12A-6Sq)(-4-6A+3Sq)+(3+6A-3Sq)(-4-6A+3Sq))\eta^5+
\end{array} \right)$$

$$\theta_1^{a2} = \left( \begin{array}{l}
\frac{3}{5}\left(4+8A-4Sq+\left(-2-3A+\frac{3Sq}{2}\right)(-4-6A+3Sq)\right)\eta^5+\frac{1}{10}(6+12A-6Sq)(3+6A-3Sq)\eta^6+\frac{2}{5}((3+6A-3Sq)\left(-2-3A+\frac{3Sq}{2}\right)+ \\
(1+2A-Sq)\begin{pmatrix}-4-6A+\\3Sq\end{pmatrix})\eta^6+\frac{2}{7}(3+6A-3Sq)(1+2A-Sq)\eta^7)+\frac{1}{Re}(-9A^2\eta^2-3A(-4-6A+3Sq)\eta^2-3(2A+A^2(-4-6A+3Sq))\eta^3- \\
(A(6+12A-6Sq)+(-4-6A+3Sq)(1+A(-4-6A+3Sq)))\eta^3-\frac{3}{2}\left(1+A^2(3+6A-3Sq)+2A\left(-2-3A+\frac{3Sq}{2}\right)+2A(-4-6A+3Sq)\right)\eta^4- \\
\frac{1}{2}((-4-6A+3Sq)\left(-6-9A+A(3+6A-3Sq)+\frac{9Sq}{2}\right)+(6+12A-6Sq)(1+A(-4-6A+3Sq)))\eta^4-\frac{9}{10}(-4-6A+2A(3+6A-3Sq)+ \\
2A(1+2A-Sq)+3Sq+(-4-6A+3Sq)\left(1+2A\left(-2-3A+\frac{3Sq}{2}\right)\right))\eta^5-\frac{3}{10}((6+12A-6Sq)\left(-6-9A+A(3+6A-3Sq)+\frac{9Sq}{2}\right)+ \\
(-4-6A+3Sq)\left(4+8A-4Sq+\left(-2-3A+\frac{3Sq}{2}\right)(-4-6A+3Sq)\right))\eta^5-\frac{3}{5}(2+4A-2Sq+\left(-2-3A+\frac{3Sq}{2}\right)^2+ \\
(-4-6A+3Sq)(-4-6A+2A\begin{pmatrix}1+2A-\\Sq\end{pmatrix}+3Sq)+\begin{pmatrix}3+6A-\\3Sq\end{pmatrix}\left(1+2A\left(-2-3A+\frac{3Sq}{2}\right)\right))\eta^6-\frac{1}{5}((-4-6A+3Sq)\begin{pmatrix}(3+6A-3Sq)\left(-2-3A+\frac{3Sq}{2}\right)+\\(1+2A-Sq)(-4-6A+3Sq)\end{pmatrix}+ \\
(6+12A-6Sq)\left(4+8A-4Sq+\left(-2-3A+\frac{3Sq}{2}\right)(-4-6A+3Sq)\right))\eta^6-\frac{3}{7}(2(1+2A-Sq)\left(-2-3A+\frac{3Sq}{2}\right)+ \\
(3+6A-3Sq)(-4-6A+2A(1+2A-Sq)+3Sq)+(-4-6A+3Sq)(2+4A-2Sq+\left(-2-3A+\frac{3Sq}{2}\right)^2))\eta^7-\frac{1}{7}(\begin{pmatrix}3+6A-\\3Sq\end{pmatrix}(1+2A-Sq)\begin{pmatrix}-4-6A+\\3Sq\end{pmatrix}+ \\
(6+12A-6Sq)\left((3+6A-3Sq)\left(-2-3A+\frac{3Sq}{2}\right)+(1+2A-Sq)(-4-6A+3Sq)\right))\eta^7-\frac{3}{28}(6+12A-6Sq)(3+6A-3Sq)(1+2A-Sq)\eta^8- \\
\frac{9}{28}((1+2A-Sq)^2+2(1+2A-Sq)\left(-2-3A+\frac{3Sq}{2}\right)(-4-6A+3Sq)+(3+6A-3Sq)\left(2+4A-2Sq+\left(-2-3A+\frac{3Sq}{2}\right)^2\right))\eta^8- \\
\frac{1}{4}(2(3+6A-3Sq)(1+2A-Sq)\left(-2-3A+\frac{3Sq}{2}\right)+(1+2A-Sq)^2\begin{pmatrix}-4-6A+\\3Sq\end{pmatrix})\eta^9-\frac{1}{5}(3+6A-3Sq)(1+2A-Sq)^2\eta^{10})+\frac{1}{2}Sq(3\eta^2+3A\begin{pmatrix}-4-6A+\\3Sq\end{pmatrix}\eta^2+ \\
(-4-6A+A(6+12A-6Sq)+3Sq)\eta^3+\begin{pmatrix}-8-12A+\\6Sq\end{pmatrix}\eta^3+\frac{1}{2}\left(6+12A-6Sq+\left(-2-3A+\frac{3Sq}{2}\right)(-4-6A+3Sq)\right)\eta^4+\frac{1}{2}\left(6+12A-6Sq+\begin{pmatrix}-4-6A+\\3Sq\end{pmatrix}^2\right)\eta^4+ \\
\frac{3}{5}(3+6A-3Sq)(-4-6A+3Sq)\eta^5+\frac{3}{10}\left((6+12A-6Sq)\left(-2-3A+\frac{3Sq}{2}\right)+(1+2A-Sq)(-4-6A+3Sq)\right)\eta^5+\frac{1}{5}(3+6A-3Sq)^2\eta^6+ \\
\frac{1}{5}(6+12A-6Sq)(1+2A-Sq)\eta^6+(6+12A-6Sq)\left(\frac{1}{6}(-4-6A+3Sq)\eta^3+\frac{1}{12}(6+12A-6Sq)\eta^4\right))-3\gamma(\frac{1}{4}(-4-6A+3Sq)^2\eta^2+ \\
\frac{1}{12}(2(6+12A-6Sq)(-4-6A+3Sq)+(-4-6A+3Sq)^3)\eta^3+\frac{1}{24}((6+12A-6Sq)^2+2(6+12A-6Sq)(-4-6A+3Sq)^2
\end{array} \right)$$

$$\theta_1^{b} = \left( \begin{array}{l}
(3+6A-3Sq)(-4-6A+3Sq)^2)\eta^4+\frac{1}{40}((6+12A-6Sq)^2(-4-6A+3Sq)+2(6+12A-6Sq)(3+6A-3Sq)(-4-6A+3Sq))\eta^5+ \\
\frac{1}{60}(6+12A-6Sq)^2(3+6A-3Sq)\eta^6+\frac{1}{Re}6(\frac{A^2\eta^2}{2}+\frac{1}{6}(2A+A^2(-4-6A+3Sq))\eta^3+\frac{1}{12}(1+A^2(3+6A-3Sq)+2A\left(-2-3A+\frac{3Sq}{2}\right)+ \\
2A(-4-6A+3Sq))\eta^4+\frac{1}{20}(-4-6A+2A(3+6A-3Sq)+2A(1+2A-Sq)+3Sq+(-4-6A+3Sq)\left(1+2A\left(-2-3A+\frac{3Sq}{2}\right)\right))\eta^5+ \\
\frac{1}{30}(2+4A-2Sq+\left(-2-3A+\frac{3Sq}{2}\right)^2+(-4-6A+3Sq)(-4-6A+2A(1+2A-Sq)+3Sq)+(3+6A-3Sq)\left(1+2A\left(-2-3A+\frac{3Sq}{2}\right)\right))\eta^6+ \\
\frac{1}{42}(2\begin{pmatrix}1+2A-\\Sq\end{pmatrix}\left(-2-3A+\frac{3Sq}{2}\right)+(3+6A-3Sq)(-4-6A+2A\begin{pmatrix}1+2A-\\Sq\end{pmatrix}+3Sq)+\begin{pmatrix}-4-6A+\\3Sq\end{pmatrix}\left(2+4A-2Sq+\left(-2-3A+\frac{3Sq}{2}\right)^2\right))\eta^7+ \\
\frac{1}{56}((1+2A-Sq)^2+2(1+2A-Sq)\left(-2-3A+\frac{3Sq}{2}\right)(-4-6A+3Sq)+(3+6A-3Sq)\left(2+4A-2Sq+\left(-2-3A+\frac{3Sq}{2}\right)^2\right))\eta^8+ \\
\frac{1}{72}(2(3+6A-3Sq)(1+2A-Sq)\left(-2-3A+\frac{3Sq}{2}\right)+(1+2A-Sq)^2(-4-6A+3Sq))\eta^9+\frac{1}{90}(3+6A-3Sq)(1+2A-Sq)^2\eta^{10}))+ \\
\beta(6A(2+2A(-4-6A+3Sq))\eta^2+2(A(-24-36A+2A(6+12A-6Sq)+18Sq)+(1+A(-4-6A+3Sq))(2+2A(-4-6A+3Sq)))\eta^3+ \\
((-24-36A+2A(6+12A-6Sq)+18Sq)(1+A(-4-6A+3Sq))+\left(-6-9A+A(3+6A-3Sq)+\frac{9Sq}{2}\right)(2+2A(-4-6A+3Sq))+ \\
A(24+48A-24Sq+2\left(-2-3A+\frac{3Sq}{2}\right)(-4-6A+3Sq)+2(-4-6A+3Sq)^2))\eta^4+ \\
\frac{3}{5}(\left(-6-9A+A(3+6A-3Sq)+\frac{9Sq}{2}\right)(-24-36A+2A(6+12A-6Sq)+18Sq)+A(2(6+12A-6Sq)\left(-2-3A+\frac{3Sq}{2}\right)+ \\
4(3+6A-3Sq)(-4-6A+3Sq)+2(1+2A-Sq)(-4-6A+3Sq))+(2+2A(-4-6A+3Sq))(4+8A-4Sq+ \\
\left(-2-3A+\frac{3Sq}{2}\right)(-4-6A+3Sq))+(1+A(-4-6A+3Sq))4Sq+\left(-2-3A+\frac{3Sq}{2}\right)(-4-6A+3Sq))+(1+A(-4-6A+3Sq))
\end{array} \right)$$

$$\theta_1^{c1} = \left( \begin{array}{l}
\frac{2}{5}(A(2(3+6A-3Sq)^2+2(6+12A-6Sq)(1+2A-Sq))+(2+2A(-4-6A+3Sq))((3+6A-3Sq)(-2-3A+\frac{3Sq}{2})+(1+2A-Sq)(-4-6A+3Sq))+ \\
(1+A(-4-6A+3Sq))(2(6+12A-6Sq)(-2-3A+\frac{3Sq}{2})+4(3+6A-3Sq)(-4-6A+3Sq)+2(1+2A-Sq)(-4-6A+3Sq))+ \\
(-24-36A+2A(6+12A-6Sq)+18Sq)(4+8A-4Sq+(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq))+(-6-9A+A(3+6A-3Sq)+\frac{9Sq}{2}) \\
(24+48A-24Sq+2(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq)+2(-4-6A+3Sq)^2))\eta^6+\frac{2}{7}((2(3+6A-3Sq)^2+2(6+12A-6Sq)(1+2A-Sq)) \\
(1+A(-4-6A+3Sq))+(3+6A-3Sq)(1+2A-Sq)(2+2A(-4-6A+3Sq))+(-24-36A+2A(6+12A-6Sq)+18Sq)((3+6A-3Sq)(-2-3A+\frac{3Sq}{2})+ \\
(1+2A-Sq)(-4-6A+3Sq))+(-6-9A+A(3+6A-3Sq)+\frac{9Sq}{2})(2(6+12A-6Sq)(-2-3A+\frac{3Sq}{2})+4(3+6A-3Sq)(-4-6A+3Sq)+2(1+2A-Sq)(-4-6A+3Sq))+ \\
(4+8A-4Sq+(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq))(24+48A-24Sq+2(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq)+2(-4-6A+3Sq)^2))\eta^7+\frac{3}{14}(\begin{pmatrix} 2(3+6A-3Sq)^2+ \\ 2(6+12A-6Sq)(1+2A-Sq) \end{pmatrix} \\
(-6-9A+A(3+6A-3Sq)+\frac{9Sq}{2})+(3+6A-3Sq)(1+2A-Sq)(-24-36A+2A(6+12A-6Sq)+18Sq)+(2(6+12A-6Sq)(-2-3A+\frac{3Sq}{2})+4\begin{pmatrix} 3+6A- \\ 3Sq \end{pmatrix}\begin{pmatrix} -4-6A+ \\ 3Sq \end{pmatrix}+ \\
2(1+2A-Sq)(-4-6A+3Sq))(4+8A-4Sq+(-2-3A+\frac{3Sq}{2})-4-6A+3Sq))+((3+6A-3Sq)(-2-3A+\frac{3Sq}{2})+1+2A-Sq)(-4-6A+3Sq))(24+48A-24Sq+ \\
2(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq)+2(-4-6A+3Sq)^2))\eta^8+\frac{1}{6}(((3+6A-3Sq)(-2-3A+\frac{3Sq}{2})+(1+2A-Sq)(-4-6A+3Sq))(2(6+12A-6Sq)(-2-3A+\frac{3Sq}{2})+ \\
4(3+6A-3Sq)(-4-6A+3Sq)+2(1+2A-Sq)(-4-6A+3Sq))+(2(3+6A-3Sq)^2+2(6+12A-6Sq)(1+2A-Sq))(4+8A-4Sq+(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq))+ \\
(3+6A-3Sq)(1+2A-Sq)(24+48A-24Sq+2(-2-3A+\frac{3Sq}{2})(-4-6A+3Sq)+2(-4-6A+3Sq)^2))\eta^9+\frac{2}{15}((2(3+6A-3Sq)^2+2(6+12A-6Sq)(1+2A-Sq)) \\
((3+6A-3Sq)(-2-3A+\frac{3Sq}{2})+(1+2A-Sq)(-4-6A+3Sq))+(3+6A-3Sq)(1+2A-Sq)(2(6+12A-6Sq)(-2-3A+\frac{3Sq}{2})+4(3+6A-3Sq)(-4-6A+3Sq)
\end{array} \right)$$

and

$$\theta_1^{c2} = \left( \begin{array}{l}
2(1+2A-Sq)(-4-6A+3Sq)))\eta^{10}+\frac{6}{55}(2(3+6A-3Sq)^2+2(6+12A-6Sq)(1+2A-Sq))(3+6A-3Sq)(1+2A-Sq)\eta^{11}+\frac{1}{\mathrm{Re}}72(\frac{A^3\eta^2}{2}+\frac{1}{6}(3A^2+A^3(-4-6A+3Sq))\eta^3+ \\
\frac{1}{12}(2A+A^3(3+6A-3Sq)+A^2(-2-3A+\frac{3Sq}{2})+3A^2(-4-6A+3Sq)+A(1+2A(-2-3A+\frac{3Sq}{2})))\eta^4+\frac{1}{20}(1+3A^2(3+6A-3Sq)+A^2(1+2A-Sq)+4A(-2-3A+\frac{3Sq}{2})+ \\
A(-4-6A+2A(1+2A-Sq)+3Sq)+(-4-6A+3Sq)(2A+A^2(-2-3A+\frac{3Sq}{2})+A(1+2A(-2-3A+\frac{3Sq}{2}))))\eta^5+\frac{1}{30}(-4-6A+4A(1+2A-Sq)+3Sq+ \\
(-2-3A+\frac{3Sq}{2})(1+2A(-2-3A+\frac{3Sq}{2}))+A(2+4A-2Sq+(-2-3A+\frac{3Sq}{2})^2)+(-4-6A+3Sq)(1+A^2(1+2A-Sq)+4A(-2-3A+\frac{3Sq}{2})+ \\
A(-4-6A+2A(1+2A-Sq)+3Sq))+(3+6A-3Sq)(2A+A^2(-2-3A+\frac{3Sq}{2})+A(1+2A(-2-3A+\frac{3Sq}{2}))))\eta^6+\frac{1}{42}(2+4A-2Sq+2A(1+2A-Sq)(-2-3A+\frac{3Sq}{2})+ \\
(-2-3A+\frac{3Sq}{2})^2+(-2-3A+\frac{3Sq}{2})(-4-6A+2A(1+2A-Sq)+3Sq)+1+2A-Sq)(1+2A(-2-3A+\frac{3Sq}{2}))+(3+6A-3Sq)(1+A^2(1+2A-Sq)+4A(-2-3A+\frac{3Sq}{2})+ \\
A(-4-6A+2A(1+2A-Sq)+3Sq))+(-4-6A+3Sq)(-4-6A+4A(1+2A-Sq)+3Sq+(-2-3A+\frac{3Sq}{2})(1+2A(-2-3A+\frac{3Sq}{2}))+A(2+4A-2Sq+(-2-3A+\frac{3Sq}{2})^2)))\eta^7+ \\
\frac{1}{56}(A(1+2A-Sq)^2+2(1+2A-Sq)(-2-3A+\frac{3Sq}{2})+(1+2A-Sq)(-4-6A+2A(1+2A-Sq)+3Sq)+(-2-3A+\frac{3Sq}{2})(2+4A-2Sq+(-2-3A+\frac{3Sq}{2})^2)+ \\
(-4-6A+3Sq)(2+4A-2Sq+2A(1+2A-Sq)(-2-3A+\frac{3Sq}{2})+(-2-3A+\frac{3Sq}{2})^2+(-2-3A+\frac{3Sq}{2})(-4-6A+2A(1+2A-Sq)+3Sq)+(1+2A-Sq)(1+2A(-2-3A+\frac{3Sq}{2})))
\end{array} \right)$$

$$
\theta_1^d = \left(\begin{aligned}
&(3+6A-3Sq)(-4-6A+4A(1+2A-Sq)+3Sq+\left(-2-3A+\frac{3Sq}{2}\right)\left(1+2A\left(-2-3A+\frac{3Sq}{2}\right)\right)+A\left(2+4A-2Sq+\left(-2-3A+\frac{3Sq}{2}\right)^2\right)))\eta^8+\\
&\frac{1}{72}((1+2A-Sq)^2+2(1+2A-Sq)\left(-2-3A+\frac{3Sq}{2}\right)^2+(1+2A-Sq)\left(2+4A-2Sq+\left(-2-3A+\frac{3Sq}{2}\right)^2\right)+(3+6A-3Sq)\left(-2-3A+\frac{3Sq}{2}\right)^2+\\
&\left(-2-3A+\frac{3Sq}{2}\right)(-4-6A+2A(1+2A-Sq)+3Sq)+(1+2A-Sq)\left(1+2A\left(-2-3A+\frac{3Sq}{2}\right)\right))+(-4-6A+3Sq)(A(1+2A-Sq)^2+\\
&2(1+2A-Sq)\left(-2-3A+\frac{3Sq}{2}\right)+(1+2A-Sq)(-4-6A+2A(1+2A-Sq)+3Sq)+\left(-2-3A+\frac{3Sq}{2}\right)\left(2+4A-2Sq+\left(-2-3A+\frac{3Sq}{2}\right)^2\right)))\eta^9+\\
&\frac{1}{90}(3(1+2A-Sq)^2\left(-2-3A+\frac{3Sq}{2}\right)+(-4-6A+3Sq)((1+2A-Sq)^2+2(1+2A-Sq)\left(-2-3A+\frac{3Sq}{2}\right)^2+(1+2A-Sq)\left(2+4A-2Sq+\left(-2-3A+\frac{3Sq}{2}\right)^2\right))+\\
&(3+6A-3Sq)(A(1+2A-Sq)^2+2(1+2A-Sq)\left(-2-3A+\frac{3Sq}{2}\right)+(1+2A-Sq)(-4-6A+2A(1+2A-Sq)+3Sq)+\\
&(-2-3A+\frac{3Sq}{2})\left(2+4A-2Sq+\left(-2-3A+\frac{3Sq}{2}\right)^2\right)))\eta^{10}+\frac{1}{110}((1+2A-Sq)^3+3(1+2A-Sq)^2\left(-2-3A+\frac{3Sq}{2}\right)(-4-6A+3Sq)+(3+6A-3Sq)((1+2A-Sq)^2+\\
&2(1+2A-Sq)\left(-2-3A+\frac{3Sq}{2}\right)^2+(1+2A-Sq)\left(\begin{aligned}&2+4A-2Sq+\\&\left(-2-3A+\frac{3Sq}{2}\right)^2\end{aligned}\right)))\eta^{11}+\frac{1}{132}(3(3+6A-3Sq)(1+2A-Sq)^2\left(-2-3A+\frac{3Sq}{2}\right)+\\
&(1+2A-Sq)^3(-4-6A+3Sq))\eta^{12}+\frac{1}{156}(3+6A-3Sq)(1+2A-Sq)^3\eta^{13}))))+\Pr(\frac{Sq\eta^3\gamma_1\gamma_2}{12(\gamma_1+\gamma_2+\gamma_1\gamma_2)}-\frac{\left(\frac{A\eta^2}{2}+\frac{\eta^3}{6}+\frac{1}{12}\left(-2-3A+\frac{3Sq}{2}\right)\eta^4+\frac{1}{20}(1+2A-Sq)\eta^5\right)\gamma_1\gamma_2}{\gamma_1+\gamma_2+\gamma_1\gamma_2}))
\end{aligned}\right)
$$

Also, Equations (49), (40) and (51) are solved using the corresponding boundary conditions in Equations (54b), (55b) and (56b) but the solutions are too big to be included in this paper.
Adopting the definition in Equation (35), the solution of the Equation (23) is given as

$$\theta(\eta) = \theta_0 + \theta_1 + \ldots \tag{62}$$

## 4. Numerical Procedure for the analysis of the governing equation

Equation (22) and (23) are nonlinear ordinary differential equations which are in this work are analyzed numerically using fifth-order Runge-Kutta Fehlberg method (Cash-Karp Runge-Kutta) coupled with shooting method, the fourth-order and second-order ordinary differential equations are decomposed into a system of first-order differential equations as follows:

$$f' = p, \tag{63}$$

$$f'' = p' = q, \tag{64}$$

$$f''' = q' = w, \tag{65}$$

$$f^{iv} = w' = z, \tag{66}$$

$$
\begin{aligned}
z = {} & fw - \frac{Sq}{2}(3q - \eta w) - \left(M^2 + \frac{1}{Da}\right)q + \alpha\left(-2qw - pz - fz' + \frac{Sq}{2}(5z - \eta z')\right) \\
& + \gamma(2qw + pz) + \beta\left(7q^2 + 24pqw + 3p^2 z + \mathrm{Re}\left(3qw^2 + \frac{3}{2}q^2 z\right)\right) = 0,
\end{aligned} \tag{67}
$$

From Equation (67), we established that

$$z' = \frac{\left\{ \begin{array}{l} fw - \frac{Sq}{2}\left(3q - \eta w\right) - \left(M^2 + \frac{1}{Da}\right)q + \alpha\left(-2qw - pz + \frac{5zSq}{2}\right) - \gamma\left(2qw + pz\right) \\ +\beta\left(7q^2 + 24pqw + 3p^2 z + \mathrm{Re}\left(3qw^2 + \frac{3}{2}q^2 z\right)\right) - z \end{array} \right\}}{\alpha\left(f + \frac{Sq\eta}{2}\right)} \tag{68}$$

The above Equations (63)-(66) and (66) can be written as

$$a\left(\eta, f, p, q, w, z\right) = p, \tag{69}$$

$$b\left(\eta, f, p, q, w, z\right) = q, \tag{70}$$

$$c\left(\eta, f, p, q, w, z\right) = w, \tag{71}$$

$$d\left(\eta, f, p, q, w, z\right) = z \tag{72}$$

$$e\left(\eta, f, p, q, z\right) = \frac{\left\{ \begin{array}{l} fw - \frac{Sq}{2}\left(3q - \eta w\right) - \left(M^2 + \frac{1}{Da}\right)q + \alpha\left(-2qw - pz + \frac{5zSq}{2}\right) - \gamma\left(2qw + pz\right) \\ +\beta\left(7q^2 + 24pqw + 3p^2 z + \mathrm{Re}\left(3qw^2 + \frac{3}{2}q^2 z\right)\right) - z \end{array} \right\}}{\alpha\left(f + \frac{Sq\eta}{2}\right)} \tag{73}$$

The iterative scheme of the fifth-order Runge-Kutta Fehlberg method (Cash-Karp Runge-Kutta) for the above system of first-order equations is given as

$$f_{i+1} = f_i + h\left(\frac{2835}{27648}k_1 + \frac{18575}{48384}k_3 + \frac{13525}{55296}k_4 + \frac{277}{14336}k_5 + \frac{1}{4}k_6\right) \tag{74}$$

$$p_{i+1} = p_i + h\left(\frac{2835}{27648}l_1 + \frac{18575}{48384}l_3 + \frac{13525}{55296}l_4 + \frac{277}{14336}l_5 + \frac{1}{4}l_6\right) \tag{75}$$

$$q_{i+1} = q_i + h\left(\frac{2835}{27648}m_1 + \frac{18575}{48384}m_3 + \frac{13525}{55296}m_4 + \frac{277}{14336}m_5 + \frac{1}{4}m_6\right) \tag{76}$$

$$w_{i+1} = w_i + h\left(\frac{2835}{27648}n_1 + \frac{18575}{48384}n_3 + \frac{13525}{55296}n_4 + \frac{277}{14336}n_5 + \frac{1}{4}n_6\right) \tag{77}$$

$$z_{i+1} = z_i + h\left(\frac{2835}{27648}r_1 + \frac{18575}{48384}r_3 + \frac{13525}{55296}r_4 + \frac{277}{14336}r_5 + \frac{1}{4}r_6\right) \tag{78}$$

where

$$k_1 = a\left(\eta_i, f_i, p_i, q_i, w_i, z_i\right)$$

$$l_1 = b\left(\eta_i, f_i, p_i, q_i, w_i, z_i\right)$$

$$m_1 = c\left(\eta_i, f_i, p_i, q_i, w_i, z_i\right)$$

$$n_1 = d\left(\eta_i, f_i, p_i, q_i, w_i, z_i\right)$$

$$r_1 = e\left(\eta_i, f_i, p_i, q_i, w_i, z_i\right)$$

$$k_2 = a\left(\eta_i + \frac{1}{5}h,\ f_i + \frac{1}{5}k_1 h,\ p_i + \frac{1}{5}l_1 h,\ q_i + \frac{1}{5}m_1 h,\ w_i + \frac{1}{5}n_1 h,\ z_i + \frac{1}{5}r_1 h\right)$$

$$l_2 = b\left(\eta_i + \frac{1}{5}h,\ f_i + \frac{1}{5}k_1 h,\ p_i + \frac{1}{5}l_1 h,\ q_i + \frac{1}{5}m_1 h,\ w_i + \frac{1}{5}n_1 h,\ z_i + \frac{1}{5}r_1 h\right)$$

$$m_2 = c\left(\eta_i + \frac{1}{5}h,\ f_i + \frac{1}{5}k_1 h,\ p_i + \frac{1}{5}l_1 h,\ q_i + \frac{1}{5}m_1 h,\ w_i + \frac{1}{5}n_1 h,\ z_i + \frac{1}{5}r_1 h\right)$$

$$n_2 = d\left(\eta_i + \frac{1}{5}h,\ f_i + \frac{1}{5}k_1 h,\ p_i + \frac{1}{5}l_1 h,\ q_i + \frac{1}{5}m_1 h,\ w_i + \frac{1}{5}n_1 h,\ z_i + \frac{1}{5}r_1 h\right)$$

$$r_2 = e\left(\eta_i + \frac{1}{5}h,\ f_i + \frac{1}{5}k_1 h,\ p_i + \frac{1}{5}l_1 h,\ q_i + \frac{1}{5}m_1 h,\ w_i + \frac{1}{5}n_1 h,\ z_i + \frac{1}{5}r_1 h\right)$$

$$k_3 = a\left(\begin{array}{l} \eta_i + \frac{3}{10}h,\ f_i + \frac{3}{40}k_1 h + \frac{9}{40}k_2 h,\ \ p_i + \frac{3}{40}l_1 h + \frac{9}{40}l_2 h, \\ q_i + \frac{3}{40}m_1 h + \frac{9}{40}m_2 h,\ \ w_i + \frac{3}{40}n_1 h + \frac{9}{40}n_2 h,\ z_i + \frac{3}{40}r_1 h + \frac{9}{40}r_2 h \end{array}\right)$$

$$l_3 = b\left(\begin{array}{l} \eta_i + \frac{3}{10}h,\ f_i + \frac{3}{40}k_1 h + \frac{9}{40}k_2 h,\ \ p_i + \frac{3}{40}l_1 h + \frac{9}{40}l_2 h, \\ q_i + \frac{3}{40}m_1 h + \frac{9}{40}m_2 h,\ \ w_i + \frac{3}{40}n_1 h + \frac{9}{40}n_2 h,\ z_i + \frac{3}{40}r_1 h + \frac{9}{40}r_2 h \end{array}\right)$$

$$m_3 = c\left(\begin{array}{l} \eta_i + \frac{3}{10}h,\ f_i + \frac{3}{40}k_1 h + \frac{9}{40}k_2 h,\ \ p_i + \frac{3}{40}l_1 h + \frac{9}{40}l_2 h, \\ q_i + \frac{3}{40}m_1 h + \frac{9}{40}m_2 h,\ \ w_i + \frac{3}{40}n_1 h + \frac{9}{40}n_2 h,\ z_i + \frac{3}{40}r_1 h + \frac{9}{40}r_2 h \end{array}\right)$$

$$n_3 = d\left(\begin{array}{l} \eta_i + \frac{3}{10}h,\ f_i + \frac{3}{40}k_1 h + \frac{9}{40}k_2 h,\ \ p_i + \frac{3}{40}l_1 h + \frac{9}{40}l_2 h, \\ q_i + \frac{3}{40}m_1 h + \frac{9}{40}m_2 h,\ \ w_i + \frac{3}{40}n_1 h + \frac{9}{40}n_2 h,\ z_i + \frac{3}{40}r_1 h + \frac{9}{40}r_2 h \end{array}\right)$$

$$r_3 = e\begin{pmatrix} \eta_i + \frac{3}{10}h,\ f_i + \frac{3}{40}k_1h + \frac{9}{40}k_2h,\ \ p_i + \frac{3}{40}l_1h + \frac{9}{40}l_2h, \\ q_i + \frac{3}{40}m_1h + \frac{9}{40}m_2h,\ \ w_i + \frac{3}{40}n_1h + \frac{9}{40}n_2h,\ z_i + \frac{3}{40}r_1h + \frac{9}{40}r_2h \end{pmatrix}$$

$$k_4 = a\begin{pmatrix} \eta_i + \frac{3}{5}h,\ f_i + \frac{3}{10}k_1h - \frac{9}{10}k_2h + \frac{6}{5}k_3h,\ \ p_i + \frac{3}{10}l_1h - \frac{9}{10}l_2h + \frac{6}{5}l_3h, \\ q_i + \frac{3}{10}m_1h - \frac{9}{10}m_2h + \frac{6}{5}m_3h,\ w_i + \frac{3}{10}n_1h - \frac{9}{10}n_2h + \frac{6}{5}n_3h, \\ z_i + \frac{3}{10}r_1h - \frac{9}{10}r_2h + \frac{6}{5}r_3h \end{pmatrix}$$

$$l_4 = b\begin{pmatrix} \eta_i + \frac{3}{5}h,\ f_i + \frac{3}{10}k_1h - \frac{9}{10}k_2h + \frac{6}{5}k_3h,\ \ p_i + \frac{3}{10}l_1h - \frac{9}{10}l_2h + \frac{6}{5}l_3h, \\ q_i + \frac{3}{10}m_1h - \frac{9}{10}m_2h + \frac{6}{5}m_3h,\ w_i + \frac{3}{10}n_1h - \frac{9}{10}n_2h + \frac{6}{5}n_3h, \\ z_i + \frac{3}{10}r_1h - \frac{9}{10}r_2h + \frac{6}{5}r_3h \end{pmatrix}$$

$$m_4 = c\begin{pmatrix} \eta_i + \frac{3}{5}h,\ f_i + \frac{3}{10}k_1h - \frac{9}{10}k_2h + \frac{6}{5}k_3h,\ \ p_i + \frac{3}{10}l_1h - \frac{9}{10}l_2h + \frac{6}{5}l_3h, \\ q_i + \frac{3}{10}m_1h - \frac{9}{10}m_2h + \frac{6}{5}m_3h,\ w_i + \frac{3}{10}n_1h - \frac{9}{10}n_2h + \frac{6}{5}n_3h, \\ z_i + \frac{3}{10}r_1h - \frac{9}{10}r_2h + \frac{6}{5}r_3h \end{pmatrix}$$

$$n_4 = d\begin{pmatrix} \eta_i + \frac{3}{5}h,\ f_i + \frac{3}{10}k_1h - \frac{9}{10}k_2h + \frac{6}{5}k_3h,\ \ p_i + \frac{3}{10}l_1h - \frac{9}{10}l_2h + \frac{6}{5}l_3h, \\ q_i + \frac{3}{10}m_1h - \frac{9}{10}m_2h + \frac{6}{5}m_3h,\ w_i + \frac{3}{10}n_1h - \frac{9}{10}n_2h + \frac{6}{5}n_3h, \\ z_i + \frac{3}{10}r_1h - \frac{9}{10}r_2h + \frac{6}{5}r_3h \end{pmatrix}$$

$$r_4 = e\begin{pmatrix} \eta_i + \frac{3}{5}h,\ f_i + \frac{3}{10}k_1h - \frac{9}{10}k_2h + \frac{6}{5}k_3h,\ \ p_i + \frac{3}{10}l_1h - \frac{9}{10}l_2h + \frac{6}{5}l_3h, \\ q_i + \frac{3}{10}m_1h - \frac{9}{10}m_2h + \frac{6}{5}m_3h,\ w_i + \frac{3}{10}n_1h - \frac{9}{10}n_2h + \frac{6}{5}n_3h, \\ z_i + \frac{3}{10}r_1h - \frac{9}{10}r_2h + \frac{6}{5}r_3h \end{pmatrix}$$

$$k_5 = a\begin{pmatrix} \eta_i + h,\ f_i - \frac{11}{54}k_1h + \frac{5}{2}k_2h - \frac{70}{27}k_3h + \frac{35}{27}k_4h,\ \ p_i - \frac{11}{54}l_1h + \frac{5}{2}l_2h - \frac{70}{27}l_3h + \frac{35}{27}l_4h, \\ q_i - \frac{11}{54}m_1h + \frac{5}{2}m_2h - \frac{70}{27}m_3h + \frac{35}{27}m_4h,\ \ w_i - \frac{11}{54}n_1h + \frac{5}{2}n_2h - \frac{70}{27}n_3h + \frac{35}{27}n_4h, \\ z_i - \frac{11}{54}r_1h + \frac{5}{2}r_2h - \frac{70}{27}r_3h + \frac{35}{27}r_4h \end{pmatrix}$$

$$l_5 = b\begin{pmatrix} \eta_i + h,\ f_i - \frac{11}{54}k_1h + \frac{5}{2}k_2h - \frac{70}{27}k_3h + \frac{35}{27}k_4h,\ \ p_i - \frac{11}{54}l_1h + \frac{5}{2}l_2h - \frac{70}{27}l_3h + \frac{35}{27}l_4h, \\ q_i - \frac{11}{54}m_1h + \frac{5}{2}m_2h - \frac{70}{27}m_3h + \frac{35}{27}m_4h,\ \ w_i - \frac{11}{54}n_1h + \frac{5}{2}n_2h - \frac{70}{27}n_3h + \frac{35}{27}n_4h, \\ z_i - \frac{11}{54}r_1h + \frac{5}{2}r_2h - \frac{70}{27}r_3h + \frac{35}{27}r_4h \end{pmatrix}$$

$$m_5 = c\begin{pmatrix} \eta_i + h,\ f_i - \frac{11}{54}k_1h + \frac{5}{2}k_2h - \frac{70}{27}k_3h + \frac{35}{27}k_4h,\ \ p_i - \frac{11}{54}l_1h + \frac{5}{2}l_2h - \frac{70}{27}l_3h + \frac{35}{27}l_4h, \\ q_i - \frac{11}{54}m_1h + \frac{5}{2}m_2h - \frac{70}{27}m_3h + \frac{35}{27}m_4h,\ \ w_i - \frac{11}{54}n_1h + \frac{5}{2}n_2h - \frac{70}{27}n_3h + \frac{35}{27}n_4h, \\ z_i - \frac{11}{54}r_1h + \frac{5}{2}r_2h - \frac{70}{27}r_3h + \frac{35}{27}r_4h \end{pmatrix}$$

$$n_5 = c\begin{pmatrix} \eta_i + h,\ f_i - \frac{11}{54}k_1h + \frac{5}{2}k_2h - \frac{70}{27}k_3h + \frac{35}{27}k_4h,\ \ p_i - \frac{11}{54}l_1h + \frac{5}{2}l_2h - \frac{70}{27}l_3h + \frac{35}{27}l_4h, \\ q_i - \frac{11}{54}m_1h + \frac{5}{2}m_2h - \frac{70}{27}m_3h + \frac{35}{27}m_4h,\ \ w_i - \frac{11}{54}n_1h + \frac{5}{2}n_2h - \frac{70}{27}n_3h + \frac{35}{27}n_4h, \\ z_i - \frac{11}{54}r_1h + \frac{5}{2}r_2h - \frac{70}{27}r_3h + \frac{35}{27}r_4h \end{pmatrix}$$

$$r_5 = d\begin{pmatrix} \eta_i + h,\ f_i - \frac{11}{54}k_1h + \frac{5}{2}k_2h - \frac{70}{27}k_3h + \frac{35}{27}k_4h,\ \ p_i - \frac{11}{54}l_1h + \frac{5}{2}l_2h - \frac{70}{27}l_3h + \frac{35}{27}l_4h, \\ q_i - \frac{11}{54}m_1h + \frac{5}{2}m_2h - \frac{70}{27}m_3h + \frac{35}{27}m_4h,\ \ w_i - \frac{11}{54}n_1h + \frac{5}{2}n_2h - \frac{70}{27}n_3h + \frac{35}{27}n_4h, \\ z_i - \frac{11}{54}r_1h + \frac{5}{2}r_2h - \frac{70}{27}r_3h + \frac{35}{27}r_4h \end{pmatrix}$$

$$k_6 = a\begin{pmatrix} \eta_i + \frac{7}{8}h,\ f_i + \frac{1631}{55296}k_1h + \frac{175}{512}k_2h + \frac{575}{13824}k_3h + \frac{44275}{110592}k_4h + \frac{253}{4096}k_5h, \\ p_i + \frac{1631}{55296}l_1h + \frac{175}{512}l_2h + \frac{575}{13824}l_3h + \frac{44275}{110592}l_4h + \frac{253}{4096}l_5h, \\ q_i + \frac{1631}{55296}m_1h + \frac{175}{512}m_2h + \frac{575}{13824}m_3h + \frac{44275}{110592}m_4h + \frac{253}{4096}m_5h, \\ n_i + \frac{1631}{55296}w_1h + \frac{175}{512}w_2h + \frac{575}{13824}w_3h + \frac{44275}{110592}w_4h + \frac{253}{4096}w_5h, \\ z_i + \frac{1631}{55296}r_1h + \frac{175}{512}r_2h + \frac{575}{13824}r_3h + \frac{44275}{110592}r_4h + \frac{253}{4096}r_5h \end{pmatrix}$$

$$l_6 = b\begin{pmatrix} \eta_i + \frac{7}{8}h,\ f_i + \frac{1631}{55296}k_1h + \frac{175}{512}k_2h + \frac{575}{13824}k_3h + \frac{44275}{110592}k_4h + \frac{253}{4096}k_5h, \\ p_i + \frac{1631}{55296}l_1h + \frac{175}{512}l_2h + \frac{575}{13824}l_3h + \frac{44275}{110592}l_4h + \frac{253}{4096}l_5h, \\ q_i + \frac{1631}{55296}m_1h + \frac{175}{512}m_2h + \frac{575}{13824}m_3h + \frac{44275}{110592}m_4h + \frac{253}{4096}m_5h, \\ n_i + \frac{1631}{55296}w_1h + \frac{175}{512}w_2h + \frac{575}{13824}w_3h + \frac{44275}{110592}w_4h + \frac{253}{4096}w_5h, \\ z_i + \frac{1631}{55296}r_1h + \frac{175}{512}r_2h + \frac{575}{13824}r_3h + \frac{44275}{110592}r_4h + \frac{253}{4096}r_5h \end{pmatrix}$$

$$m_6 = c\begin{pmatrix} \eta_i + \frac{7}{8}h,\ f_i + \frac{1631}{55296}k_1h + \frac{175}{512}k_2h + \frac{575}{13824}k_3h + \frac{44275}{110592}k_4h + \frac{253}{4096}k_5h, \\ p_i + \frac{1631}{55296}l_1h + \frac{175}{512}l_2h + \frac{575}{13824}l_3h + \frac{44275}{110592}l_4h + \frac{253}{4096}l_5h, \\ q_i + \frac{1631}{55296}m_1h + \frac{175}{512}m_2h + \frac{575}{13824}m_3h + \frac{44275}{110592}m_4h + \frac{253}{4096}m_5h, \\ n_i + \frac{1631}{55296}w_1h + \frac{175}{512}w_2h + \frac{575}{13824}w_3h + \frac{44275}{110592}w_4h + \frac{253}{4096}w_5h, \\ z_i + \frac{1631}{55296}r_1h + \frac{175}{512}r_2h + \frac{575}{13824}r_3h + \frac{44275}{110592}r_4h + \frac{253}{4096}r_5h \end{pmatrix}$$

$$n_6 = d\begin{pmatrix} \eta_i + \frac{7}{8}h,\ f_i + \frac{1631}{55296}k_1h + \frac{175}{512}k_2h + \frac{575}{13824}k_3h + \frac{44275}{110592}k_4h + \frac{253}{4096}k_5h, \\ p_i + \frac{1631}{55296}l_1h + \frac{175}{512}l_2h + \frac{575}{13824}l_3h + \frac{44275}{110592}l_4h + \frac{253}{4096}l_5h, \\ q_i + \frac{1631}{55296}m_1h + \frac{175}{512}m_2h + \frac{575}{13824}m_3h + \frac{44275}{110592}m_4h + \frac{253}{4096}m_5h, \\ n_i + \frac{1631}{55296}w_1h + \frac{175}{512}w_2h + \frac{575}{13824}w_3h + \frac{44275}{110592}w_4h + \frac{253}{4096}w_5h, \\ z_i + \frac{1631}{55296}r_1h + \frac{175}{512}r_2h + \frac{575}{13824}r_3h + \frac{44275}{110592}r_4h + \frac{253}{4096}r_5h \end{pmatrix}$$

$$r_6 = e\begin{pmatrix} \eta_i + \frac{7}{8}h,\ f_i + \frac{1631}{55296}k_1h + \frac{175}{512}k_2h + \frac{575}{13824}k_3h + \frac{44275}{110592}k_4h + \frac{253}{4096}k_5h, \\ p_i + \frac{1631}{55296}l_1h + \frac{175}{512}l_2h + \frac{575}{13824}l_3h + \frac{44275}{110592}l_4h + \frac{253}{4096}l_5h, \\ q_i + \frac{1631}{55296}m_1h + \frac{175}{512}m_2h + \frac{575}{13824}m_3h + \frac{44275}{110592}m_4h + \frac{253}{4096}m_5h, \\ n_i + \frac{1631}{55296}w_1h + \frac{175}{512}w_2h + \frac{575}{13824}w_3h + \frac{44275}{110592}w_4h + \frac{253}{4096}w_5h, \\ z_i + \frac{1631}{55296}r_1h + \frac{175}{512}r_2h + \frac{575}{13824}r_3h + \frac{44275}{110592}r_4h + \frac{253}{4096}r_5h \end{pmatrix}$$

Using the above fifth-order Runge-Kutta Fehlberg method coupled with shooting method, computer programs are written in MATLAB for the solutions of the Equation (14). The results for step size, $h = 0.01$ are presented in the following section.

## 5. Parameters of Engineering Interest

Another set of important considerations in the analysis of fluid flow and heat transfer are skin friction coefficient and Nusselt number. These are some of the parameters of engineering interest in the fluid and heat transfer study.

The local skin friction coefficient at lower disk is

$$C_f = \frac{\tau_w|_{x=0}}{\frac{1}{2}\rho U_w^2} \tag{79}$$

while the local Nusselt number at the disk is

$$Nu = \frac{h(\mathrm{t})q_w}{K\left(T_f - T_h\right)}\Big|_{z=0} \tag{80}$$

where wall heat flux is defined as:

$$q_w |_{z=0} = -K \frac{\partial T}{\partial z} |_{z=0} + q_r |_{z=0} \tag{81}$$

Using the similarity variables in Equation (14) and the dimensionless parameters in Equation (26), one obtains the dimensionless forms of the local skin friction coefficient and Nusselt number as

$$C_f = \sqrt{2}\,\mathrm{Re}^{-0.5} \left( \begin{array}{c} 2f''(0) + \alpha\left(3Sqf''(0) - 2Af'''(0) + 2f''(0)\right) \\ -2\gamma f''(0) + \beta\left(\dfrac{\mathrm{Re}}{4} f'''^{3}(0) + 6f''(0)\right) \end{array} \right) \tag{82}$$

$$Nu = \left(1 + Rd\right)\theta'(0) \tag{83}$$

## 6. Results and Discussion

The results of homotopy perturbation method and the developed fifth-order Runge-Kutta Fehlberg method (RKFM) coupled with shooting method are presented in Table 3. Parametric studies are carried out and the influences of various parameters on the flow and heat transfer processes are established as shown in Figs. 2-18.

Table 3: Numerical values of skin friction for different parameter

| $\alpha$ | $\gamma$ | $\beta$ | $Re$ | $M$ | $S_q$ | $A$ | RKFM $-Re^{\frac{1}{2}}C_f$ | HAM [39] $-Re^{\frac{1}{2}}C_f$ | HPM $-Re^{\frac{1}{2}}C_f$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 0.10 | 0.10 | 1.00 | 0.40 | 1.00 | 0.01 | 3.88243 | 3.88243 | 3.88243 |
| 0.02 | | | | | | | 3.95950 | 3.95950 | 3.95950 |
| 0.03 | | | | | | | 4.03655 | 4.03655 | 4.03655 |
| 0.01 | 0.11 | | | | | | 3.85249 | 3.85249 | 3.85249 |
| | 0.12 | | | | | | 3.82255 | 3.82255 | 3.82255 |
| | 0.13 | | | | | | 3.79260 | 3.79260 | 3.79260 |
| | 0.10 | 0.20 | | | | | 4.63884 | 4.63884 | 4.63884 |
| | | 0.25 | | | | | 5.02298 | 5.02298 | 5.02298 |
| | | 0.29 | | | | | 5.33196 | 5.33196 | 5.33196 |
| | | 0.10 | 0.10 | | | | 3.85868 | 3.85868 | 3.85868 |
| | | | 0.20 | | | | 3.86116 | 3.86116 | 3.86116 |
| | | | 0.30 | | | | 3.86368 | 3.86368 | 3.86368 |
| | | | 1.00 | 0.50 | | | 3.90088 | 3.90088 | 3.90088 |
| | | | | 0.60 | | | 3.92337 | 3.92337 | 3.92337 |
| | | | | 0.70 | | | 3.94985 | 3.94985 | 3.94985 |
| | | | | 0.40 | 0.70 | | 6.41631 | 6.41631 | 6.41631 |
| | | | | | 0.75 | | 6.00413 | 6.00413 | 6.00413 |
| | | | | | 0.80 | | 5.58953 | 5.58953 | 5.58953 |
| | | | | | 1.00 | 0.02 | 4.06568 | 4.06568 | 4.06568 |
| | | | | | | 0.03 | 4.24817 | 4.24817 | 4.24817 |
| | | | | | | 0.04 | 4.42993 | 4.42993 | 4.42993 |

Table 4: Numerical values of Nusselt number for different parameter

| | | | | | | | | | RKFM | HAM [39] | HPM |
|---|---|---|---|---|---|---|---|---|---|---|---|
| α | γ | β | Sq | Pr | Ec | $\gamma_1$ | $\gamma_2$ | Rd | *Nu* | *Nu* | *Nu* |
| 0.01 | 0.10 | 0.10 | 1.00 | 0.10 | 0.10 | 0.20 | 0.20 | 0.20 | 0.092034 | 0.092034 | 0.092034 |
| 0.02 | | | | | | | | | 0.091945 | 0.091945 | 0.091945 |
| 0.03 | | | | | | | | | 0.091856 | 0.091856 | 0.091856 |
| 0.01 | 0.11 | | | | | | | | 0.092149 | 0.092149 | 0.092149 |
| | 0.12 | | | | | | | | 0.092265 | 0.092265 | 0.092265 |
| | 0.13 | | | | | | | | 0.092380 | 0.092380 | 0.092380 |
| | 0.10 | 0.20 | | | | | | | 0.088806 | 0.088806 | 0.088806 |
| | | 0.25 | | | | | | | 0.087186 | 0.087186 | 0.087186 |
| | | 0.29 | | | | | | | 0.085563 | 0.085563 | 0.085563 |
| | | 0.10 | 0.70 | | | | | | 0.093319 | 0.093319 | 0.093319 |
| | | | 0.75 | | | | | | 0.092502 | 0.092502 | 0.092502 |
| | | | 0.80 | | | | | | 0.093528 | 0.093528 | 0.093528 |
| | | | 1.00 | 0.11 | | | | | 0.090331 | 0.090331 | 0.090331 |
| | | | | 0.12 | | | | | 0.088628 | 0.088628 | 0.088628 |
| | | | | 0.13 | | | | | 0.086926 | 0.086926 | 0.086926 |
| | | | | 0.10 | 0.15 | | | | 0.083306 | 0.083306 | 0.083306 |
| | | | | | 0.25 | | | | 0.065851 | 0.065851 | 0.065851 |
| | | | | | 0.35 | | | | 0.048396 | 0.048396 | 0.048396 |
| | | | | | 0.10 | 0.30 | | | 0.108540 | 0.108540 | 0.108540 |
| | | | | | | 0.40 | | | 0.119230 | 0.119230 | 0.119230 |
| | | | | | | 0.50 | | | 0.126720 | 0.126720 | 0.126720 |
| | | | | | | 0.20 | 0.70 | | 0.090240 | 0.090240 | 0.090240 |
| | | | | | | | 0.80 | | 0.092731 | 0.092731 | 0.092731 |
| | | | | | | | 0.90 | | 0.094174 | 0.094174 | 0.094174 |
| | | | | | | | 0.20 | 0.30 | 0.101120 | 0.101120 | 0.101120 |
| | | | | | | | | 0.40 | 0.110210 | 0.110210 | 0.110210 |
| | | | | | | | | 0.50 | 0.119300 | 0.119300 | 0.119300 |

## 6.1 Influence of the third-grade fluid parameters on dimensionless velocity profile

Figs. 1-4 depict the influence of fluid flow parameters on dimensionless velocity profile. It is evident that an increase in the fluid flow parameters of the squeezing flow causes a corresponding increase in the fluid flow velocity. This is because the fluid flow parameters in question vary inversely with the viscosity of the fluid being squeezed. As these parameters increase, the viscosity of the fluid decreases and consequently increases the velocity of the fluid as the molecules of the squeezing flow are free to move with less restriction. These parameters can be used as a monitoring agent as they directly affect the viscosity of the third grade squeezing fluid.

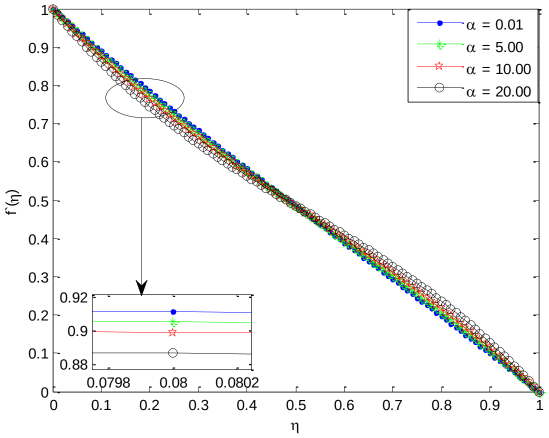


**Figure. 2** Influence of α on dimensionless velocity profile

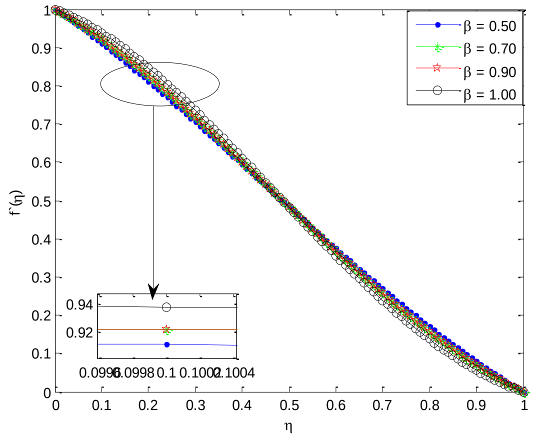


**Figure. 3** Influence of β on dimensionless velocity profile

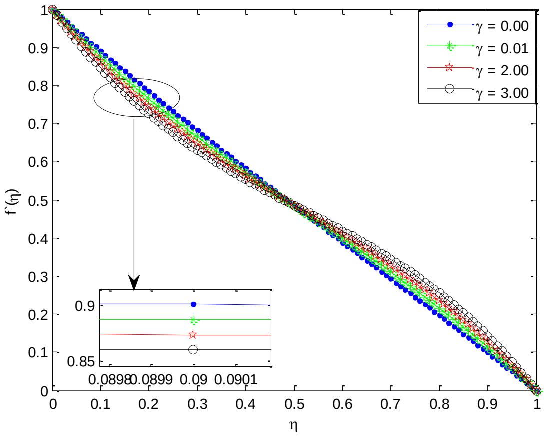


**Figure. 4** Influence of γ on dimensionless velocity profile

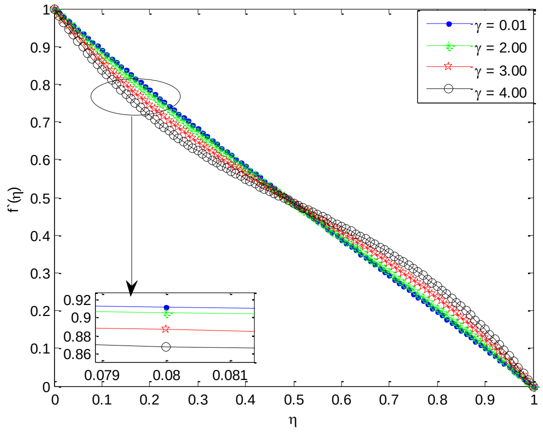


**Figure. 5** Influence of γ on dimensionless velocity profile

## 6.2 Influence of suction and squeezing parameters on dimensionless velocity profile

Figs. 6-7 depict the Influence of suction and squeezing parameters on dimensionless velocity profile. Figure 6 shows how the suction parameter affects the velocity of the squeezing discs for an increasing value of the fluid parameters. It is obvious that for a suction parameter greater than zero, the radial velocity of the lower disc increases while that of the upper disc decreases as a result of a corresponding increase in the viscosity of the third-grade fluid from the lower squeezing disc to the upper disc. Fig. 7 depicts the impact of the squeezing parameter on the fluid flow between the parallel discs. The figure shows that an increase in the squeezing parameter causes a corresponding increase in the squeezing rate. This is because as the squeezing parameter increases, the radial velocity of the squeezing discs increases there by generating a driving compressive rotary force on the fluid flowing between the two parallel discs.

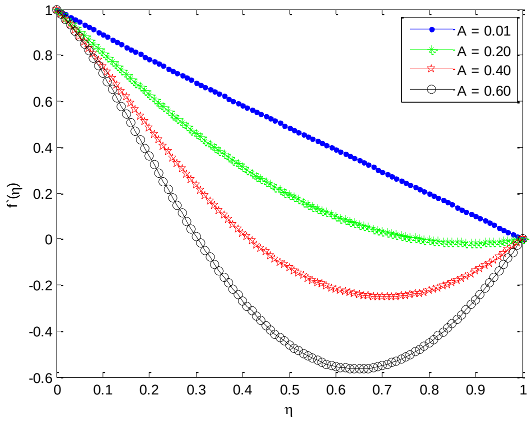


Figure. 6 Influence of suction term on dimensionless velocity profile

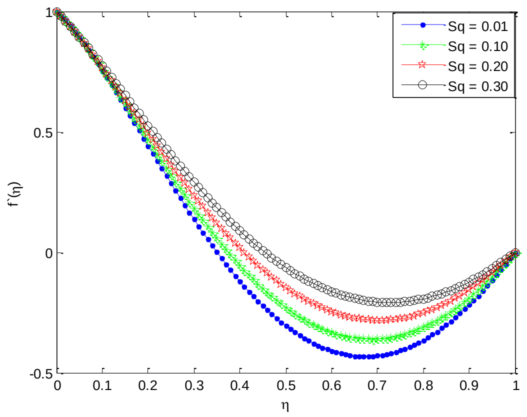


Figure. 7 Influence of squeezing term on dimensionless velocity profile

## 6.3 Influence of Hartman and Reynold's number on dimensionless velocity profile

Figs. 8-9 depict the influence of Hartman number and Reynold's number on dimensionless velocity profile. Figure 8 shows how the Hartman number affects the velocity of the squeezing discs for an increasing value of the fluid parameters. It is obvious that for a Hartman number greater than zero or for an increasing Hartman number, the radial velocity of the lower disc decreases while that of the upper disc increases as a result of a corresponding increase in the viscosity of the third-grade fluid from the upper disc to the lower squeezing disc. As the Hartman number becomes large, it automatically raises the magnetic field as a result of a corresponding increase in the Lorentz force, hence decreases the flow velocity while increase in Reynold's number increases the velocity of the fluid as shown in Fig. 9.

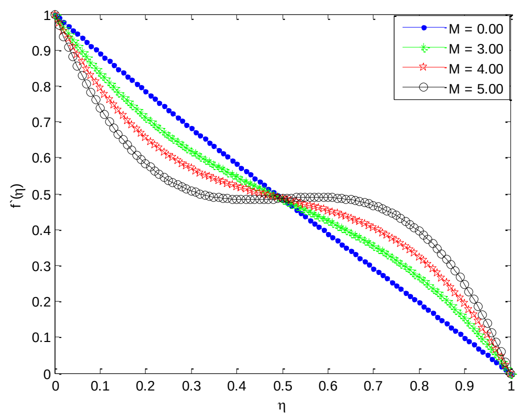


**Figure. 8** Influence of Hartman number on dimensionless velocity profile.

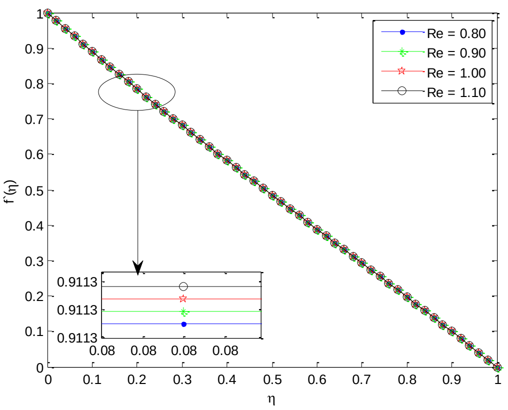


**Figure. 9** Influence of Reynold's number on dimensionless velocity profile.

## 6.4 Influence of the third-grade fluid parameter and Reynold's number on dimensionless temperature profile

Figs. 10-11 depict the influence of the third-grade fluid parameter and Reynold's number on dimensionless temperature profile. It has been ascertained that an increase in the third-grade fluid parameter causes reduction in the fluid viscosity thereby increasing resistance between the fluid molecules. However, as the Reynold's number associated with the third-grade fluid increases, a decreasing effect is noticed in the dimensionless temperature profile. This is because there is a reduction in the convective capability of a high velocity fluid as compared to that with a moderate velocity.

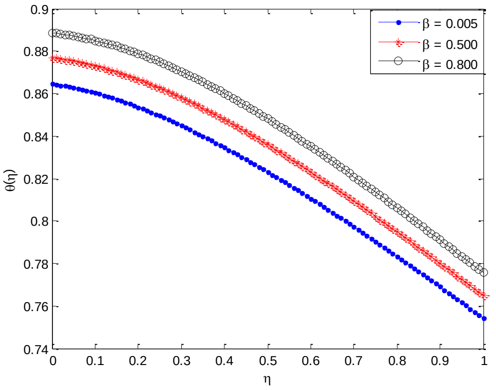


**Figure. 10** Influence of the third grade fluid parameter on Dimensionless temperature profile

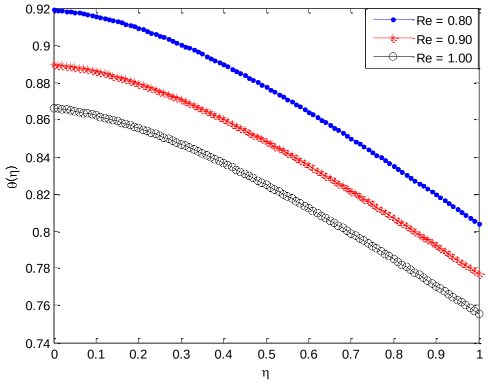


**Figure. 11** Influence of Reynold's number on dimensionless Dimensionless temperature profile

## 6.5 Influence of Prandtl and Eckert number on dimensionless temperature profile

Figs. 12-13 depict the influence of Prandtl and Eckert number on dimensionless temperature profile. Figure 12 depicts a decrease in temperature profile as the Prandtl number increases. This is because an increase in the Prandtl number reduces thermal diffusivity thereby reducing the temperature profile. In Fig. 13, Eckert number is observed to have a linear increasing property on the dimensionless temperature profile. This is because of the increase in the total kinetic energy of the fluid which correspondingly elevate the fluid temperature.

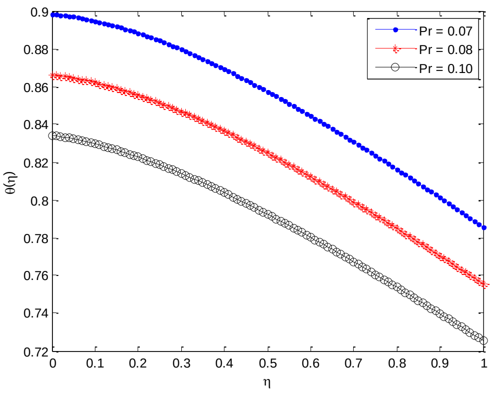


**Figure. 12** Influence of Prandtl number on dimensionless temperature profile

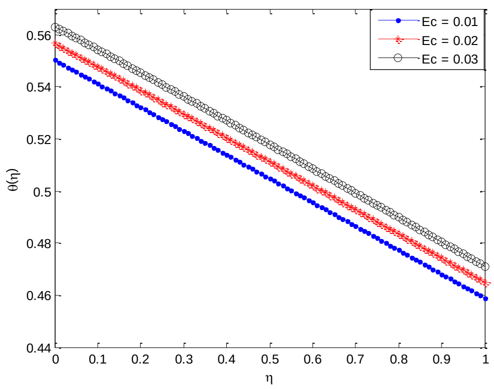


**Figure. 13** Influence of Eckert number on dimensionless temperature profile

## 6.6 Influence of Hartman number and Squeezing parameter on dimensionless temperature profile

Figs. 14-15 depict the influence of Hartman number and Squeezing parameter on dimensionless temperature profile. It is clear that as the Hartman number becomes large, it automatically raises the magnetic field as a result of an increase in the Lorentz force. This makes the temperature profile to increase as the Hartman number increases. Considering Fig. 15, a rapid increase in the dimensionless temperature profile is noticed for a large value of squeezing parameter as a result of a driving compressive rotary force which generates a noticeable heating effect thereby increasing the temperature profile.

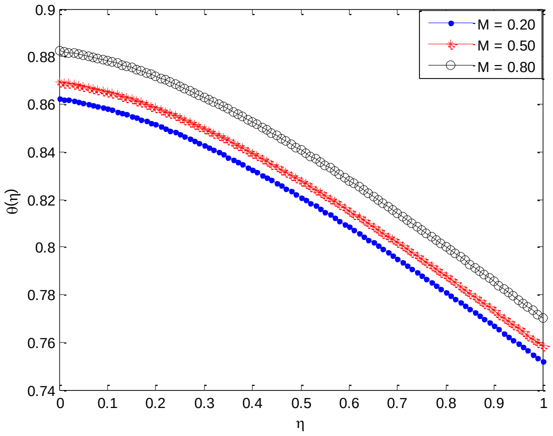


**Figure. 14** Influence of Hartman number on dimensionless temperature profile

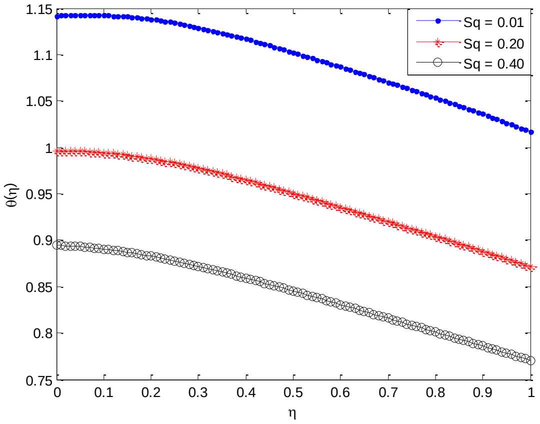


**Figure. 15** Influence of squeezing parameter on dimensionless temperature profile

## 6.7 Influence of the thermal Biot number and Radiation term on dimensionless temperature profile

Figures 16-18 depict the influence of the thermal Biot number and Radiation term on dimensionless temperature profile. In Figures 16-17, the two thermal Biot number have opposing effect on the dimensionless temperature profile, but the cooling effect generated by the first Biot number is more that the temperature rising effect obtained from the second even for the same range of values. As a result, these parameters can serve as a control for temperature monitoring. However, in Figure 18, as the radiation parameter increases, the dimensionless temperature profile increases. This is because, an increase in the radiation property causes a reduction in the absorptivity and consequently increases the rate of heat transfer to the third-grade fluid.

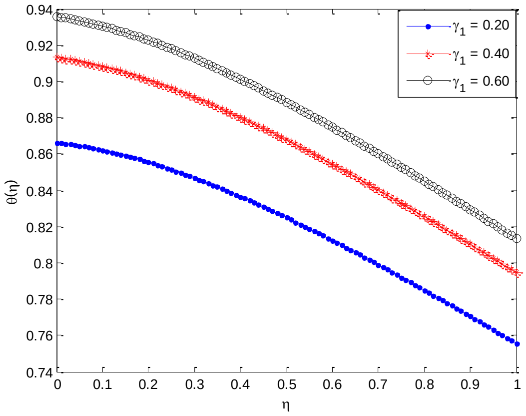


**Figure. 16** Influence of the first thermal Biot number on dimensionless temperature profile

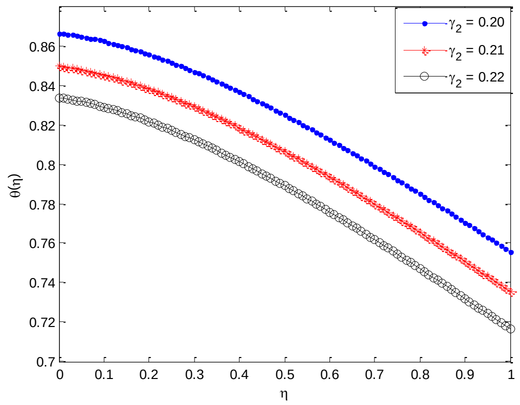


**Figure. 17** Influence of the second thermal Biot number on dimensionless temperature profile

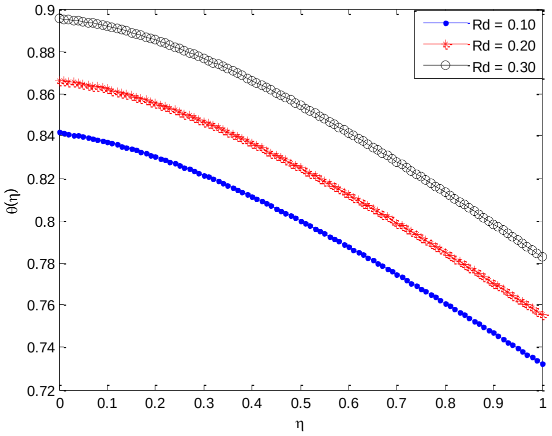


**Figure. 18** Influence of Radiation term on dimensionless temperature profile

## 7. Conclusion

In this study, nonlinear analysis of unsteady squeezing flow and heat transfer of a third grade nanofluid between two parallel disks embedded in a porous medium under the influences of thermal radiation and temperature jump boundary conditions were investigated. The developed flow and thermal models were solved analytically and numerically using homotopy perturbation method and fifth-order Runge-Kutta Fehlberg method (Cash-Karp Runge-Kutta) coupled with shooting method, respectively. Also, the influences of various flow and heat transfer parameters were investigated. Important significance of study includes the study of flow and heat transfer of third grade fluid as applied in energy conservation, coal slurries, polymer solutions, textiles, ceramics, catalytic reactors, oil recovery applications, friction reduction and micro mixing biological samples.

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## Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this research work.

## References

[1] M. Mustafa. T. Hayat and S. Obadiat, “On heat and mass transfer in the unsteady squeezing flow between parallel plates”, Mechanica, 47, 1581-1589, 2012.

[2] T. Hayat, A. Yousaf, M. Mustafa. and S. Obadiat, “MHD squeezing flow of second grade fluid between two parallel disk”, International Journal of Numerical Fluids,69, 399-410,2011.

[3] G. Domairry and A. Aziz. Approximate analysis of MHD squeeze flow between two parallel disks with suction or injection by homotopy perturbation method”, Mathematical Problems in Engineering, vol.2009, doi:1155/2009/603916.

[4] A. M. Siddiqui, S. Irum and A. R. Ansari. Unsteady squeezing flow of a viscous MHD fluid between parallel plates, Mathematical Modeling and Analysis, 13, 565-576, 2008.

[5] A. M. Rashidi, H. Shahmohamadi and S. Dinarvand. Analytic approximate solutions for unsteady two dimensional and axisymmetric squeezing flows between parallel plates", Mathematical Problems in Engineering, vol.2008, doi:10.1155/2008/935095.

[6] W. A. Khan and A. Aziz. Natural convective boundary layer flow over a vertical plate with uniform surface heat flux", International Journal of Thermal Sciences, 50, 1207-1217,2011.

[7] W. A. Khan W.A. and Aziz A. "Double diffusive natural convection boundary layer flow in a porous medium saturated with a nanofluid over a vertical plate, prescribed surface heat, solute and nanofluid fluxes, International Journal of Thermal Sciences, 50, 2154-2160,2011.

[8] A. J. Kuznetsov and N. D. Nield. "Natural convective boundary layer flow of a nanofluid past a vertical plate, International Journal of Thermal Sciences, 49, 243-247,2010.

[9] M. R. Hashimi., T. Hayat and A. Alsaedi. "On the analytic solutions for squeezing flow of nanofluids between parallel disks", Nonlinear Analysis Modeling and Control, 17(4), 418-430, 2014.

[10] C. L. M. H. Navier. "Memoir sur les lois du movement des fluids", Memoirs Academic Royal Science Institute France, 6, 389-440, 1823.

[11] C. H. Choi, J. A. Westin and K. S. Breur. "To slip or not slip water flows in hydrophilic and hydrophobic microchannels", Proceedings of IMECE 2002, New Orleans, LA, Article ID: 33707, 2002.

[12] M. T. Matthew and I. D. Boyd, "Nano boundary layer equation with nonlinear Navier boundary condition", Journal of Mathematical Analysis and Application, vol.333, pp.381-400, 2007.

[13] M. J. Martin and L. D. Boyd. Momentum and heat transfer in a laminar boundary layer with slip flow", Journal of Thermo physics and Heat Transfer, 20(4), 710-719, 2006.

[14] P. D. Ariel. Axisymmetric flow due to stretching sheet with partial slip", Computer and Mathematics with Application, 333, 381-400, 2007.

[15] C. Wang. Analysis of viscous flow due to a stretching sheet with surface slip and suction", Nonlinear Analysis:Real world application, 10(1), .375-380, 2009.

[16] K. Das. Impact of thermal radiation on MHD slip flow over a flat plate with variable fluid properties", Heat and mass transfer, 48, 767-778, 2012.

[17] K. Das. Slip effects on heat and mass transfer in MHD micro polar fluid flow over an inclined plate with thermal radiation and chemical reaction", International Journal of Numerical methods in Fluids, 70, 96-101, 2016.

[18] A. Hussain, S. T., Mohyud-Cheema and T. A. Din. Analytical and numerical approaches to squeezing flow and heat transfer between two parallel disks with velocity slip and temperature jump", China Physics Letter, 29, 1-5,2012.

[19] K. Das. Slip flow and convective heat transfer of nanofluid over a permeable stretching surface", Computers and Physics, 64(1), 34-42, 2012.

[20] R.G. Deissler. An analysis of second-order slip flow and temperature-jump boundary conditions for rarefied gases. International Journal of heat and mass transfer. Vol. 7(6),681-694, 1964.

[21] K. Das., Jana S. and Acharya N., “Slip effects on squeezing flow of nanofluid between two parallel disks”, International Journal of Applied Mechanics and Engineering, 21(1) , 5-20,2016.

[22] R. Ellahi. Hayat T., Mahomed F.M. and Asghar S. “Effects of slip on nonlinear flows of a third grade fluid”, Nonlinear Analysis: Real world Applications, 11, 139-146, 2010.

[23] S. Yao T. Fang and Y. Zhang. “Heat transfer of a generalized stretching/shrinking wall problem with convective boundary condition”, Communications in Nonlinear science and Numerical simulation, 16, 752-760, 2011.

[24] R. Kandasamy, P. Loganathanb and P. P. Arasub. Scaling group transformation for MHD boundary layer flow of a nanofluid past vertical stretching surface in the presence of suction/injection”, Nuclear Engineering and Design, 241, 2053-2059,2011.

[25] D. Makinde and A. Aziz. Boundary layer flow of a nanofluid past a stretching sheet with convective boundary conditions”, International Journal of Thermal sciences, 50, 11326-1332, 2011.

[26] M. J. Uddin, Yasser Alginahi O. Anwar Bég, M.N. Kabir. Numerical solutions for gyrotactic bioconvection in nanofluid-saturated porous media with Stefan blowing and multiple slip effects. Computers & Mathematics with Applications. Volume 72, Issue 10, November 2016, Pages 2562-2581

[27] M.J. Uddin, M.N. Kabir and Y.M. Alginahi. Lie group analysis and numerical solution of magnetohydrodynamic free convective slip flow of micropolar fluid over a moving plate with heat transfer Computers & Mathematics with Applications, Vol. 70, Issue 5, September 2015, Pages 846-856

[28] Md. Jashim, Uddin, W. A. Khan, and A. I. Md. Ismail. Scaling Group Transformation for MHD Boundary Layer Slip Flow of a Nanofluid over a Convectively Heated Stretching Sheet with Heat Generation. Mathematical Problems in Engineering, Volume 2012 (2012), ArticleID 934964,20pages.

[29] Md. Jashim Uddin, O. Anwar Bég, and Ahmad Izani Ismail. "Radiative Convective Nanofluid Flow Past a Stretching/Shrinking Sheet with Slip Effects", Journal of Thermophysics and Heat Transfer, Vol. 29, No. 3 (2015), pp. 513-523.

[30] Md. Jashim Uddin, O. A. Bég, A. Aziz, and A. I. Md. Ismail. Group Analysis of Free Convection Flow of a Magnetic Nanofluid with Chemical Reaction. Mathematical Problems in Engineering, Volume 2015 (2015), Article ID 621503, 11 pages.

[31] Md. JashimUddinO. AnwarBégMd. Nazir Uddin. Energy conversion under conjugate conduction, magneto-convection, diffusion and nonlinear radiation over a non-linearly stretching sheet with slip and multiple convective boundary conditions. Energy, Vol. 115, Part 1, 15 November 2016, Pages 1119-1129.

[32] T Zohra, T Fatema, M Jashim Uddin, M Ismail, A Izani, O Anwar Bég. Bioconvective electromagnetic nanofluid transport from a wedge geometry: Simulation of smart electro-conductive bio-nanopolymer processing. Heat Transfer—Asian Research 47 (1), 231-250.

[33] FT Zohra, MJ Uddin, AIM Ismail, OA Bég, A Kadir. Anisotropic slip magneto-bioconvection flow from a rotating cone to a nanofluid with Stefan blowing effects. Chinese Journal Of Physics-Taipei- 56(1), 2017.

[34] MJ Uddin, WA Khan, FT Zohra, AIM Ismail. Blasius and Sakiadis Slip Flows of Nanofluid with Radiation Effects. Journal of Aerospace Engineering 29 (4), 04015080

[35] K.V.S. Raju, T. Sudhakar Reddy , M.C. Raju , P.V. Satya Narayana , S. Venkataramana. MHD convective flow through porous medium in a horizontal channel with insulated and impermeable bottom wall in the presence of viscous dissipation and Joule heating. Ain Shams Engineering Journal (2014) 5, 543–551.

[36] L.Bühler, C.Mistrangelo and T.Najuch. Magnetohydrodynamic flows in model porous structures. Fusion Engineering and Design, Volumes 98–99.

[37] C. GeindreauJ. L. Auriault. Magnetohydrodynamic flow through porous media. Ecoulement magnétohydrodynamique en milieux poreux. Comptes Rendus de l'Académie des Sciences - Series IIB - Mechanics

[38] J.A. Falade , Joel C. Ukaegbu , A.C. Egere , Samuel O. Adesanya , * MHD oscillatory flow through a porous channel saturated with porous medium. Alexandria Engineering Journal (2017) 56, 147–152.

[39] J. R. Pattnaik, *, G. C. Dash, S. Singh. Radiation and mass transfer effects on MHD flow through porous medium past an exponentially accelerated inclined plate with variable temperature. Ain Shams Engineering Journal (2017) 8, 67–75.

[40] F. Ali, Khan I, S. U. Haq, S. Shafie. Influence of thermal radiation on unsteady free convection MHD flow of Brinkman type fluid in a porous medium with Newtonian heating. Hindawi Publishing Corporation, Mathematical Problems in Eng; 2013 [Article ID 632394, 13 pages].

[41] F. Ali. Khan, Khan I, A. F, S. Shafie. Effects of wall shear stress on MHD conjugate flow over an inclined plate in a porous medium with ramped wall temperature. Hindawi Publishing Corporation, Mathematical Problems in Eng; 2014.

[42] Z. Ismail, I. Khan, N.M. Nasir, Jusoh Awang R, Salleh MZ, Shafie S. The effects of magnetohydrodynamic and radiation on flow of second grade fluid past an infinite inclined plate in porous medium. In: Proceedings of AIP conference, vol. 1643; 2015. p. 563.

[43] Z. Ismail Z, A. Khan I, Mohamad AQ, Shafie S. Second grade fluid for rotating MHD of an unsteady free convection flow in a porous medium. Defect Diffusion Forum 2015;362:100–7.

[44] Z. Ismail, I. Khan, Shafie S. Rotation and heat absorption effects on unsteady MHD free convection flow in a porous medium past an infinite inclined plate with ramped wall temperature. Recent Advances in Mathematics.

[45] M. Hatami, J. Hatami, and D. D. Ganji, "Computer simulation of MHD blood conveying gold nanoparticles as a third grade non-Newtonian nanofluid in a hollow porous vessels,"Computer Method and Program in Biomedicine 113, 632–641 (2014).

[46] A. Malvandi and D. D Ganji, "Magnetic field effect on nanoparticles migration and heat transfer of water/alumina nanofluid in a channel," J. of Magn. & Magnet. Materials 362, 172–179 (2104).

[47] M. Sheikholeslami, M. Hatami, and D. D. Ganji, "Analytical investigation of MHD nanofluid flow in a semi-porous channel," Powder Technology, pages-10 (2013).

[48] M. Fakour, A. Vahabzadeh, D.D. Ganji, and M. Hatami, "Analytical study of micropolar fluid flow and heat transfer in a channel with permeable walls," J. of Mol. Liquids pages-7 (2015).

[49] M. Jashim, O. Uddin, A. Beg, and A. I. MD, "Ismail, Mathematical modeling of radiative hydromagnetic thermosolute nanofluid convection slip flow in saturated porous media," Hindawi Pub. Corp. pages-11 (2014).

[50] T. Hayat, M. U Qureshi, and Q. Hussain, "Effect of heat transfer on the peristaltic flow of an electrically conducting fluid in a porous space," Appl. Math. Modell. 33(4), 1862–1873 (2009).

[51] M Hatami, R. Nouri, and D. D. Ganji, "Forced convection analysis for MHD $Al_2O_3$-water nanofluid flow over a horizontal plate," J. of Mol. Liquids 187(0), 294–301 (2013).

[52] M. Hatami and D. D. Ganji, "Heat transfer and nanofluid flow in suction and blowing process between parallel disks in presence of variable magnetic field," J. Mol. Liquids 190(0), 159–168 (2013).

[53] M. Sheikholeslami, M. Gorji-Bandpy, R. Ellahi, M. Hassan, and S. Soleimani, "Effects of MHD on Cu water nano fluid flow and heat transfer by means of CVFEM," J. Magn.& Magnet. Mater. 349(0), 188–200 (2014).

[54] M. Sheikholeslami, M. Gorji-Bandpy, and D. D. Ganji, "Magnetic field effects on natural convection around a horizontal circular cylinder inside a square enclosure filled with nanofluid," Int. Commun. Heat Mass Transfer 39(7), 978–986 (2012).

[55] M. Sheikholeslami, M. Gorji-Bandpy, D. D. Ganji, and S. Soleimani, "Heat flux boundary condition for nanofluid filled enclosure in presence of magnetic field," J. of Mol. Liquids 193, 174–184 (2014).

[56] M. Sheikholeslami, M. G-Bandpy, D. D. Ganji, and S. Soleimani, "Effect of a magnetic field on natural convection in an enclosure between a circular and sinusoidal cylinder in the presence of magnetic field," Int. Commun. Heat Mass Transfer 39(9), 1435–1443 (2012).

[57] M Sheikholeslami and M G Bandpy, "Free convection of ferrofluid in a cavity heated from below in the presence of an external magnetic field," Powder Technology 256, 490–498 (2014).

[58] M. Sheikholeslami, K. Vajravelu, and M. M. Rashidi, "Forced convection heat transfer in a semi annulus under the influence of a variable magnetic field, International," J. of Heat and Mass Transfer 92, 339–348 (2016).

[59] M. Sheikholeslami, M. G-Bandpy, D. D. Ganji, P. Rana, and S Soleimani, “Magnetohydrodynamics free convection of Al2O3–water nanofluid considering thermophoresis and Brownian motion effects,” Computers & Fluids 94, 147–160 (2014).

[60] M. Sheikholeslami, D. D. Ganji, M. Y. Javed, and R. Ellahi, “Effect of thermal radiation on magnetohydrodynamics nanofluid flow and heat transfer by means of two phase model,” J. of Magna.& Magnet. Materials 374, 36–43 (2015).

[61] T. Hayat, T. Muhammad, A. Alsaedi, and M. S. Alhuthali, “Magnetohydrodynamics three-dimensional flow of viscoelastic nanofluid in the presence of nonlinear thermal radiation,” J. of Magn. & Magnet. Materials. 385, 222–229 (2015).

[62] T Hayat, M Rashid, M Imtiaz, and A. Alsaedi, “Magnetohydrodynamics (MHD) flow of Cu-water nanofluid due to a rotating disk with partial slip,” AIP Advances 5, 067169 (2015) doi: 10.1063/1.4923380.

[63] T. Hayat, F.M. Abbasi, M Al-Yami, and S Monaquel, “Slip and Joule heating effects in mixed convection peristaltic transport of nanofluid with Soret and Dufour effects,” J. of Mol. Liquids 194, 93–99 (2014).

[64] M.M. Rashidi, N. Vishnu Ganesh, A.K. Abdul Hakeem, and B. Ganga, “Buoyancy effect on MHD flow of nanofluid over a stretching sheet in the presence of thermal radiation,” J. of Mol. Liquids 198, 234–238 (2014).

[65] Z. Mehrez, A. ElCafsi, A. Belghith, and P. LeQuérés, “MHD effects on heat transfer and entropy generation of nanofluid flow in an open cavity,” J. of Magn. & Magnet. Materials 374, 214–224 (2015).

[66] F. Mabood, W. A. Khan, and A. I. M. Ismail, “MHD boundary layer flow and heat transfer of nanofluids over a nonlinear stretching sheet: A numerical study,” J. of Magn. & Magnet. Materials 374, 569–576 (2015).

[67] F. M. Abbasi, S.A. Shehzad, T. Hayyat, A. Alsadi, and M.A. Obid, “Influence of heat and mass flux conditions in Hydromagnetic flow of Jeffrey nanofluid,” AIP Advances 5, 037111 (2015)

[68] S.A. Shehzad, F. M. Abbasi, T. Hayyat, and A. Alsadi, “MHD Mixed convective peristaltic motion of nanofluid with joule heating and thermophoresis effects,” PLoSE ONE 9(11), (2014) doi: 10.1371/J. Pone. 0111417.

[68] M. G. Sobamowo, A. T. Akinshilo, A. A. Yinusa. “Thermo-Magneto-Solutal Squeezing Flow of Nanofluid between Two Parallel Disks Embedded in a Porous Medium: Effects of Nanoparticle Geometry, Slip, and Temperature Jump Conditions,” Hindawi Modelling and Simulation in Engineering Article ID 7364634, (2018).

[69] M. G. Sobamowo, L. O. Jayesimi and M. A. Waheed. Magnetohydrodynamic squeezing flow analysis of nanofluid under the effect of slip boundary conditions using variation of parameter method. *Karbala International Journal of Modern Science.* volume 4 (2018), 107-118.

[70] M. G. Sobamowo, A. T. Akinshilo and A. A. Yinusa. Thermo- Magneto-Solutal Squeezing Flow of Nanofluid between Two Parallel Disks Embedded in a Porous Medium: Effects of Nanoparticle Geometry, Slip, and Temperature Jump Conditions. *Modeling and Simulation in Engineering.* Volume *2018, Article ID 7364634, 18 pages.*

[71] M. G. Sobamowo, L. O. Jayesimi and M. A. Waheed. Axisymmetric Magnetohydrodynamic Squeezing flow of nanofluid in a porous medium under the influence of slip boundary conditions. *Transport Phenomena in nano and micro Scales.* volume 6 (2) (2018), 122-132.

[72] M. G. Sobamowo and A. T. Akinshilo. Analysis of flow, heat transfer and entropy generation in a pipe conveying fourth grade fluid with temperature-dependent viscosities and internal heat generation*. Journal of Molecular Liquids*, volume 241 (2018), 188-198.

[73] M. G. Sobamowo. Singular perturbation and differential transform methods to two-dimensional flow of nanofluid in a porous channel with expanding/contracting walls subjected to a uniform transverse magnetic field. *Thermal Science and Engineering Progress.* volume 4 (2017), 71-84.

[74] M. G. Sobamowo. On the analysis of laminar flow of viscous fluid through a porous channel with suction/injection at slowly expanding or contracting walls. Journal of Computational Applied Mechanics. volume 48 (2) (2017), 319-330.

[75] M. G. Sobamowo and L. O. Jayesimi. Squeezing flow analysis of nanofluid under the effects of magnetic field and slip boundary using Chebychev spectral collocation method. *Fluid Mechanics.* volume 3 (6) (2017).

[76] M. G. Sobamowo, L. O. Jayesimi and M. A. Waheed (2017). On the Squeezing flow of nanofluid through a porous medium with slip boundary and magnetic field: A comparative study of three approximate analytical methods. Global Journal of Engineering, volume 17 (6), 61-76.

[77] B. Y. Ogunmola, A. T. Akinshilo and M. G. Sobamowo. Perturbation Solutions for Hagen–Poiseuille Flow and Heat transfer of Third Grade Fluid with Temperature-Dependent Viscosities and Internal Heat Generation. Article in Press, International Journal of Engineering Mathematics, Hindawi, 2016.

[78] M. G. Sobamowo, T. A. Akishilo, A. A. Yinusa, O.A. Adedibu.: Nonlinear Slip Effects on Pipe Flow and Heat Transfer of Third Grade Fluid with Nonlinear Temperature-Dependent Viscosities and Internal Heat Generation. Software Engineering. 6(3) (2018), 69-88.

[79] A. T. Akinshilo and M. G. Sobamowo. Perturbation Solutions for the Study of MHD Blood as a Third Grade Nanofluid
Transporting Gold Nanoparticles through a Porous Channel. Journal of Applied and Computational Mechanics. volume 3 (2)(2017), 103-113.

[80] R.L. Fosdick and K.R. Rajagopal Thermodynamics and stability of fluids of third grade, *Procter Society London*, volume 339 (1980), 351-377.

[81] S.N. Majhi and V.R. Nair Flow of a third grade fluid over a sternosed tubes, *Indian national science academy*, volume 60(3) (1994), 535.

[82] M. Massoudi and I. Christie Effects of variable viscosity and viscous dissipation on the flow of a third grade fluid in a pipe, *International journal of nonlinear mechanics*, volume 30(1995) 687.

[83] M. Yurusoy M. and Pakdemirli. Approximate analytical solution for the flow of a third grade fluid in a pipe, *International journal of nonlinear mechanics*, volume 37(2002). 187-195.

[84] K. Vajrevelu, J. r. Cannon, D. Rollins and J. Leto on solutions of some non-Linear differential equations arising in third grade fluid flows, *International journal of engineering science*, volume 40(2002), 1791.

[85] T. Hayat, S. Nadeem, S. Asghar, and A. M. Siddiqui. Fluctuating flow of a third order fluid on a porous plate in a rotating medium, *International journal of nonlinear mechanics*, volume 36(2002), 901-916.

[86] Y. Muhammet. Similarity solutions to boundary layer equations for third grade non-Newtonian fluid in special channel coordinate system, *Journal of theoretical and applied mechanics*, volume 41(4) (2003) 775-787.

[87] M. Yurusoy. Flow of a third grade fluid between concentric cylinder, *Journal of mathematical and computational applications*, volume 9 (1) (2004), 11-12.

[88] M. Pakdemirli and B. S. Yilbas Entropy generation for a pipe flow of a third grade fluid with Vogel model viscosity, *International journal of nonlinear mechanics*, volume 43 (3) (2006).432-437.

[89] M. Sajid, R. Mahmood and T. Hayat (2008) Finite element solution for flow of a third grade fluid past a horizontal porous plate with partial slip, *International journal for computer and mathematics with applications*, volume 56, 1236.

[90] R. Elahi, T. Hayat, F. M. Mahomed and S. Asghar. Effects of slip on the nonlinear flow of the third grade fluid, *Journal of the nonlinear analysis*, volume 11, (2010) 139-146.

[91] O. J. Jayeoba, and S. S. Okoya . Approximate analytical solutions for pipe flow of a third grade fluid with variable model of viscosities and heat generation/absorption, *Journal of the Nigerian mathematical society*, volume 31(2012), 207-227.

[92] S. S. Abbasbandy, T. Hayat, R. Ellahi and S. Asghar. Numerical results of a flow in third grade fluid between two porous walls, *Verlagderzeitschrift fur Naturforschurg*, Volume 0932 (2008), 51.

[93] I. Nayak, A. K. Nayak and J. Padhy Numerical solutions for the flow and heat transfer of a third grade fluid past a porous vertical plate, *Journal for advanced studies theoretical physics*, volume 6, (2012), 615.

[94] Y. M. Aiyesimi, G. T. Okedayo and O. W. Lawal. Unsteady magneto hydrodynamic (MHD) thin film flow of a third grade fluid with heat transfer and no slip boundary condition down an inclined plane, *International journal of physical sciences*, volume 8(19) (2013), 946.

[95] A. W. Ogunsola and B. A. Peter. Effect of variable viscosity on third grade fluid flow over a radiative surface with Arrhenius reaction, *International journal of pure and applied sciences and technology*, Volume 22(1) (2014), 1-2.

[96] M. Yurusoy, M. Pakdemirli and B. S. Yilbas. Perturbation solution for third grade fluid flowing between parallel plates, *Proceedings of the institute of Mechanical Engineering,* Volume 222 Part C, (2007). 653.

## Appendix I

$$
\begin{aligned}
p^5 :\ & f_5'''' + f_0 f_4''' + f_1 f_3''' + f_2 f_2''' + f_3 f_1''' + f_4 f_0''' - \left(1/2\right) Sq\left(3 f_4'' + \eta f_4'''\right) - M^2 f_4'' - \alpha( f_0 f_4^{v} + f_1 f_3^{v} + f_2 f_2^{v} + \\
& f_3 f_1^{v} + f_4 f_0^{v} + 2 f_0'' f_4''' + 2 f_1'' f_3''' + 2 f_2'' f_2''' + 2 f_3'' f_1''' + 2 f_4'' f_0''' + f_0' f_4'''' + f_1' f_3'''' + f_2' f_2'''' + f_3' f_0'''' + \\
& f_4' f_0'''' - \left(1/2\right) Sq\left(5 f_4^{v} + \eta f_4^{v}\right)) - \gamma(2 f_0'' f_4''' + 2 f_1'' f_3''' + 2 f_2'' f_2''' + 2 f_3'' f_1''' + 2 f_4'' f_0''' + f_0' f_4'''' + f_1' f_3'''' + \\
& f_2' f_2'''' + f_3' f_0'''' + f_4' f_0'''') + \beta(7 f_0'' \left(2 f_0'' f_4'' + 2 f_1'' f_3'' + {f_2''}^2\right) + 7 f_1'' \left(2 f_0'' f_3'' + 2 f_1'' f_2''\right) + 7 f_2'' \left(2 f_0'' f_2'' + {f_1''}^2\right) + \\
& 14 f_3'' f_0'' f_1'' + 7 f_4'' {f_0''}^2 + 24 f_0' f_0'' f_4''' + (24( f_0' f_1'' + f_1' f_0'')) f_3''' + \left(24\left( f_0' f_2'' + f_1' f_1'' + f_2' f_0''\right)\right) f_2''' + \\
& \left(24\left( f_0' f_3'' + f_1' f_2'' + f_2' f_1'' + f_3' f_0''\right)\right) f_1''' + (24( f_0' f_4'' + f_1' f_3'' + f_2' f_2'' + f_3' f_1'' + f_4' f_0'')) f_0''' + 9 {f_0'}^2 f_4'''' + \\
& 18 f_0' f_1' f_3'''' + \left(9\left(2 f_0' f_2' + {f_1'}^2\right)\right) f_2'''' + (9(2 f_0' f_3' + 2 f_1' f_2')) f_0'''' + \left(9\left(2 f_0' f_4' + 2 f_1' f_3' + {f_2'}^2\right)\right) f_0'''' + \\
& \mathrm{Re}(3 f_0'' \left(2 f_0''' f_4''' + 2 f_1''' f_3''' + {f_2'''}^2\right) + 3 f_1'' (2 f_0''' f_3''' + 2 f_1''' f_2''') + 3 f_2'' \left(2 f_0''' f_2''' + {f_1'''}^2\right) + 6 f_3'' f_0''' f_1''' + \\
& 3 f_4'' {f_0'''}^2 + \left(3/2\right) {f_0''}^2 f_4'''' + 3 f_0'' f_1'' f_3'''' + \left(3/2\left(2 f_0'' f_2'' + {f_1''}^2\right)\right) f_2'''' + \left(3/2\left(2 f_0'' f_3'' + 2 f_1'' f_2''\right)\right) f_0'''' + \\
& \left(\frac{3}{2}\left(2 f_0'' f_4'' + 2 f_1'' f_3'' + {f_2''}^2\right)\right) f_0'''')) = 0,
\end{aligned}
$$

$$
\begin{aligned}
p^6 :\ & f_6'''' + f_0 f_5''' + f_1 f_4''' + f_2 f_3''' + f_3 f_2''' + f_4 f_1''' + f_5 f_0''' - \left(1/2\right) Sq\left(3 f_5'' + \eta f_5'''\right) - M^2 f_5'' - \alpha( f_0 f_5^{v} + f_1 f_4^{v} + \\
& f_2 f_3^{v} + f_3 f_2^{v} + f_4 f_1^{v} + f_5 f_0^{v} + 2 f_0'' f_5''' + 2 f_1'' f_4''' + 2 f_2'' f_3''' + 2 f_3'' f_2''' + 2 f_4'' f_1''' + 2 f_5'' f_0''' + f_0' f_5'''' + \\
& f_1' f_4'''' + f_2' f_3'''' + f_3' f_2'''' + f_4' f_0'''' + f_0' f_0'''' - \left(1/2\right) Sq\left(5 f_5^{v} + \eta f_5^{v}\right)) - \gamma(2 f_0'' f_5''' + 2 f_1'' f_4''' + 2 f_2'' f_3''' + \\
& 2 f_3'' f_2''' + 2 f_4'' f_1''' + 2 f_5'' f_0''' + f_0' f_5'''' + f_1' f_4'''' + f_2' f_3'''' + f_3' f_2'''' + f_4' f_0'''' + f_0' f_0'''') + \beta(7 f_0'' (2 f_0'' f_5'' + \\
& 2 f_1'' f_4'' + 2 f_2'' f_3'') + 7 f_1'' \left(2 f_0'' f_4'' + 2 f_1'' f_3'' + {f_2''}^2\right) + 7 f_2'' \left(2 f_0'' f_3'' + 2 f_1'' f_2''\right) + 7 f_3'' \left(2 f_0'' f_2'' + {f_1''}^2\right) + \\
& 14 f_4'' f_0'' f_1'' + 7 f_5'' {f_0''}^2 + 24 f_0' f_0'' f_5''' + \left(24\left( f_0' f_1'' + f_1' f_0''\right)\right) f_4''' + \left(24\left( f_0' f_2'' + f_1' f_1'' + f_2' f_0''\right)\right) f_3''' + \\
& (24( f_0' f_3'' + f_1' f_2'' + f_2' f_1'' + f_3' f_0'')) f_2''' + \left(24\left( f_0' f_4'' + f_1' f_3'' + f_2' f_2'' + f_3' f_1'' + f_4' f_0''\right)\right) f_1''' + \\
& 9 {f_0'}^2 f_5'''' + 18 f_0' f_1' f_4'''' + \left(9\left(2 f_0' f_2' + {f_1'}^2\right)\right) f_3'''' + \left(24\left( f_0' f_5'' + f_1' f_4'' + f_2' f_3'' + f_3' f_2'' + f_4' f_1'' + f_0' f_0''\right)\right) f_0''' + \\
& \left(9\left(2 f_0' f_3' + 2 f_1' f_2'\right)\right) f_2'''' + \left(9\left(2 f_0' f_4' + 2 f_1' f_3' + {f_2'}^2\right)\right) f_0'''' + \left(9\left(2 f_0' f_0' + 2 f_1' f_4' + 2 f_2' f_3'\right)\right) f_0'''' + \\
& \mathrm{Re}(3 f_0'' (2 f_0''' f_5''' + 2 f_1''' f_4''' + 2 f_2''' f_3''') + 3 f_1'' \left(2 f_0''' f_4''' + 2 f_1''' f_3''' + {f_2'''}^2\right) + 3 f_2'' \left(2 f_0''' f_3''' + 2 f_1''' f_2'''\right) + \\
& 3 f_3'' \left(2 f_0''' f_2''' + {f_1'''}^2\right) + 6 f_4'' f_0''' f_1''' + 3 f_5'' {f_0'''}^2 + \left(3/2\right) {f_0''}^2 f_5'''' + 3 f_0'' f_1'' f_4'''' + \left(3/2\left(2 f_0'' f_2'' + {f_1''}^2\right)\right) f_3'''' + \\
& \left(\frac{3}{2}\left(2 f_0'' f_3'' + 2 f_1'' f_2''\right)\right) f_2'''' + \left(3/2\left(2 f_0'' f_4'' + 2 f_1'' f_3'' + {f_2''}^2\right)\right) f_0'''' + \left(3/2\left(2 f_0'' f_5'' + 2 f_1'' f_4'' + 2 f_2'' f_3''\right)\right) f_0'''')) = 0
\end{aligned}
$$

$$
\begin{aligned}
p^7 :\ & f_7'''' + f_0 f_6''' + f_1 f_5''' + f_2 f_4''' + f_3 f_3''' + f_4 f_2''' + f_5 f_1''' + f_6 f_0''' - (1/2)Sq\left(3 f_6'' + \eta f_6'''\right) - M^2 f_6'' - \alpha( f_0 f_6^{v} + \\
& f_1 f_5^{v} + f_2 f_4^{v} + f_3 f_3^{v} + f_4 f_2^{v} + f_5 f_1^{v} + f_6 f_0^{v} + 2 f_0'' f_6''' + 2 f_1'' f_5''' + 2 f_2'' f_4''' + 2 f_3'' f_3''' + 2 f_4'' f_2''' + 2 f_5'' f_1''' + \\
& 2 f_6'' f_0''' + f_0' f_6'''' + f_1' f_5'''' + f_2' f_4'''' + f_3' f_3'''' + f_4' f_2'''' + f_0' f_0'''' + f_6' f_0'''' - (1/2)Sq\left(5 f_6^{v} + \eta f_6^{v}\right)) - \\
& \gamma(2 f_0'' f_6''' + 2 f_1'' f_5''' + 2 f_2'' f_4''' + 2 f_3'' f_3''' + 2 f_4'' f_2''' + 2 f_5'' f_1''' + 2 f_6'' f_0''' + f_0' f_6'''' + f_1' f_5'''' + f_2' f_4'''' + f_3' f_3'''' + \\
& f_4' f_2'''' + f_0' f_0'''' + f_6' f_0'''') + \beta(7 f_0''(2 f_0'' f_6'' + 2 f_1'' f_5'' + 2 f_2'' f_4'' + {f_3''}^{2}) + 7 f_1''\left(2 f_0'' f_5'' + 2 f_1'' f_4'' + 2 f_2'' f_3''\right) + \\
& 7 f_2''\left(2 f_0'' f_4'' + 2 f_1'' f_3'' + {f_2''}^{2}\right) + 7 f_3''\left(2 f_0'' f_3'' + 2 f_1'' f_2''\right) + 7 f_4''\left(2 f_0'' f_2'' + {f_1''}^{2}\right) + 14 f_5'' f_0'' f_1'' + 7 f_6'' {f_0''}^{2} + \\
& 24 f_0' f_0'' f_6''' + \left(24\left( f_0' f_1'' + f_1' f_0''\right)\right) f_5''' + \left(24\left( f_0' f_2'' + f_1' f_1'' + f_2' f_0''\right)\right) f_4''' + \left(24\left( f_0' f_3'' + f_1' f_2'' + f_2' f_1'' + f_3' f_0''\right)\right) f_3''' + \\
& (24( f_0' f_4'' + f_1' f_3'' + f_2' f_2'' + f_3' f_1'' + f_4' f_0'')) f_2''' + (24( f_0' f_5'' + f_1' f_4'' + f_2' f_3'' + f_3' f_2'' + f_4' f_1'' + f_0' f_0'')) f_1''' + \\
& (24( f_0' f_6'' + f_1' f_5'' + f_2' f_4'' + f_3' f_3'' + f_4' f_2'' + f_0' f_1'' + f_6' f_0'')) f_0''' + 9 {f_0'}^{2} f_6'''' + 18 f_0' f_1' f_5'''' + \left(9\left(2 f_0' f_2' + {f_1'}^{2}\right)\right) f_4'''' + \\
& \left(9\left(2 f_0' f_3' + 2 f_1' f_2'\right)\right) f_3'''' + \left(9\left(2 f_0' f_4' + 2 f_1' f_3' + {f_2'}^{2}\right)\right) f_2'''' + \left(9\left(2 f_0' f_0' + 2 f_1' f_4' + 2 f_2' f_3'\right)\right) f_0'''' + \\
& \left(9\left(2 f_0' f_6' + 2 f_1' f_0' + 2 f_2' f_4' + {f_3'}^{2}\right)\right) f_0'''' + \mathrm{Re}(3 f_0''\left(2 f_0''' f_6''' + 2 f_1''' f_5''' + 2 f_2''' f_4''' + {f_3'''}^{2}\right) + \\
& 3 f_1''\left(2 f_0''' f_5''' + 2 f_1''' f_4''' + 2 f_2''' f_3'''\right) + 3 f_2''(2 f_0''' f_4''' + 2 f_1''' f_3''' + {f_2'''}^{2}) + 3 f_3''\left(2 f_0''' f_3''' + 2 f_1''' f_2'''\right) + 3 f_4''\left(2 f_0''' f_2''' + {f_1'''}^{2}\right) + \\
& 6 f_5'' f_0''' f_1''' + 3 f_6'' {f_0'''}^{2} + (3/2) {f_0''}^{2} f_6'''' + 3 f_0'' f_1'' f_5'''' + \left(3/2\left(2 f_0'' f_2'' + {f_1''}^{2}\right)\right) f_4'''' + \left(3/2\left(2 f_0'' f_3'' + 2 f_1'' f_2''\right)\right) f_3'''' + \\
& \left(\frac{3}{2}\left(2 f_0'' f_4'' + 2 f_1'' f_3'' + {f_2''}^{2}\right)\right) f_2'''' + (3/2(2 f_0'' f_5'' + 2 f_1'' f_4'' + 2 f_2'' f_3'')) f_0'''' + \left(\frac{3}{2}\begin{pmatrix} 2 f_0'' f_6'' + 2 f_1'' f_5'' + \\ 2 f_2'' f_4'' + {f_3''}^{2} \end{pmatrix}\right) f_0'''')) = 0
\end{aligned}
$$

$$
\begin{aligned}
p^8 :\ & f_8'''' + f_0 f_7''' + f_1 f_6''' + f_2 f_5''' + f_3 f_4''' + f_4 f_3''' + f_5 f_2''' + f_6 f_1''' + f_7 f_0''' - (1/2)Sq\left(3 f_7'' + \eta f_7'''\right) - M^2 f_7'' - \alpha(2 f_0'' f_7''' + \\
& f_1' f_6'''' + f_5 f_2^{v} + f_0 f_7^{v} + f_2 f_5^{v} + 2 f_1'' f_6''' + 2 f_2'' f_5''' + 2 f_3'' f_4''' + 2 f_4'' f_3''' + 2 f_5'' f_2''' + 2 f_6'' f_1''' + 2 f_7'' f_0''' + f_0' f_7'''' + \\
& f_2' f_5'''' + f_3' f_4'''' + f_4' f_3'''' + f_0' f_2'''' + f_6' f_0'''' + f_7' f_0'''' + f_1 f_6^{v} + f_6 f_1^{v} + f_7 f_0^{v} + f_3 f_4^{v} + f_4 f_3^{v} - (1/2)Sq\left(5 f_7^{v} + \eta f_7^{v}\right)) - \\
& \gamma(2 f_0'' f_7''' + 2 f_1'' f_6''' + 2 f_2'' f_5''' + 2 f_3'' f_4''' + 2 f_4'' f_3''' + 2 f_5'' f_2''' + 2 f_6'' f_1''' + 2 f_7'' f_0''' + f_0' f_7'''' + f_1' f_6'''' + f_2' f_5'''' + f_3' f_4'''' + \\
& f_4' f_3'''' + f_0' f_2'''' + f_6' f_0'''' + f_7' f_0'''') + \beta(24 f_0' f_0'' f_7''' + 14 f_6'' f_0'' f_1'' + 18 f_0' f_1' f_6'''' + \mathrm{Re}(3 f_0'' \begin{pmatrix} 2 f_0''' f_7''' + 2 f_1''' f_6''' + \\ 2 f_2''' f_5''' + 2 f_3''' f_4''' \end{pmatrix} + \\
& 3 f_1''\left(2 f_0''' f_6''' + 2 f_1''' f_5''' + 2 f_2''' f_4''' + {f_3'''}^{2}\right) + 3 f_2''(2 f_0''' f_5''' + 2 f_1''' f_4''' + 2 f_2''' f_3''') + 3 f_3''\left(2 f_0''' f_4''' + 2 f_1''' f_3''' + {f_2'''}^{2}\right) + \\
& 3 f_4''\left(2 f_0''' f_3''' + 2 f_1''' f_2'''\right) + 3 f_5''\left(2 f_0''' f_2''' + {f_1'''}^{2}\right) + 6 f_6'' f_0''' f_1''' + 3 f_7'' {f_0'''}^{2} + (3/2) {f_0''}^{2} f_7'''' + 3 f_0'' f_1'' f_6'''' + \\
& \left(3/2\left(2 f_0'' f_2'' + {f_1''}^{2}\right)\right) f_5'''' + \left(3/2\left(2 f_0'' f_3'' + 2 f_1'' f_2''\right)\right) f_4'''' + \left(3/2\left(2 f_0'' f_4'' + 2 f_1'' f_3'' + {f_2''}^{2}\right)\right) f_3'''' + \\
& \left(3/2\left(2 f_0'' f_5'' + 2 f_1'' f_4'' + 2 f_2'' f_3''\right)\right) f_2'''' + (3/2(2 f_0'' f_6'' + 2 f_1'' f_5'' + 2 f_2'' f_4'' + {f_3''}^{2})) f_0'''' + (3/2(2 f_0'' f_7'' + 2 f_1'' f_6'' + \\
& 2 f_2'' f_5'' + 2 f_3'' f_4'')) f_0'''') + 7 f_0''(2 f_0'' f_7'' + 2 f_1'' f_6'' + 2 f_2'' f_5'' + 2 f_3'' f_4'') + \left(9\left(2 f_0' f_2' + {f_1'}^{2}\right)\right) f_5'''' + 9 {f_0'}^{2} f_7'''' + \\
& 7 f_1''\left(2 f_0'' f_6'' + 2 f_1'' f_5'' + 2 f_2'' f_4'' + {f_3''}^{2}\right) + 7 f_3''\left(2 f_0'' f_4'' + 2 f_1'' f_3'' + {f_2''}^{2}\right) + 7 f_5''\left(2 f_0'' f_2'' + {f_1''}^{2}\right) + \\
& 7 f_4''\left(2 f_0'' f_3'' + 2 f_1'' f_2''\right) + \left(24\left( f_0' f_3'' + f_1' f_2'' + f_2' f_1'' + f_3' f_0''\right)\right) f_4''' + 7 f_7'' {f_0''}^{2} + 7 f_2''\left(2 f_0'' f_5'' + 2 f_1'' f_4'' + 2 f_2'' f_3''\right) + \\
& (24( f_0' f_7'' + f_1' f_6'' + f_2' f_5'' + f_3' f_4'' + f_4' f_3'' + f_0' f_2'' + f_6' f_1'' + f_7' f_0'')) f_0''' + (24( f_0' f_6'' + f_1' f_5'' + f_2' f_4'' + f_3' f_3'' + \\
& f_4' f_2'' + f_0' f_1'' + f_6' f_0'')) f_1''' + \left(9\left(2 f_0' f_3' + 2 f_1' f_2'\right)\right) f_4'''' + \left(9\left(2 f_0' f_4' + 2 f_1' f_3' + {f_2'}^{2}\right)\right) f_3'''' + \left(9\left(2 f_0' f_0' + 2 f_1' f_4' + 2 f_2' f_3'\right)\right) f_2'''' + \\
& \left(24\left( f_0' f_5'' + f_1' f_4'' + f_2' f_3'' + f_3' f_2'' + f_4' f_1'' + f_0' f_0''\right)\right) f_2''' + \left(24\left( f_0' f_2'' + f_1' f_1'' + f_2' f_0''\right)\right) f_5''' + (24( f_0' f_4'' + f_1' f_3'' + f_2' f_2'' + \\
& f_3' f_1'' + f_4' f_0'')) f_3''' + \left(24\left( f_0' f_1'' + f_1' f_0''\right)\right) f_6''' + \left(9\left(2 f_0' f_7' + 2 f_1' f_6' + 2 f_2' f_0' + 2 f_3' f_4'\right)\right) f_0'''' + \left(9\begin{pmatrix} 2 f_0' f_6' + 2 f_1' f_0' + \\ 2 f_2' f_4' + {f_3'}^{2} \end{pmatrix}\right) f_0'''') = 0
\end{aligned}
$$

$$
\begin{aligned}
p^9 :\ & f_9'''' + f_0 f_8''' + f_1 f_7''' + f_2 f_6''' + f_3 f_5''' + f_4 f_4''' + f_5 f_3''' + f_6 f_2''' + f_7 f_1''' + f_8 f_0''' - (1/2)Sq(3f_8'' + \eta f_8''') - M^2 f_8'' - \alpha( f_4 f_4^{v} + f_1 f_7^{v} + \\
& f_2 f_6^{v} + f_0 f_8^{v} + f_7 f_1^{v} + f_8 f_0^{v} + f_6 f_2^{v} + f_3 f_5^{v} + f_5 f_3^{v} + 2f_1'' f_7''' + 2f_2'' f_6''' + 2f_3'' f_5''' + 2f_4'' f_4''' + 2f_5'' f_3''' + 2f_6'' f_2''' + 2f_7'' f_1''' + \\
& 2f_8'' f_0''' + f_0' f_8'''' + f_2' f_6'''' + f_3' f_5'''' + f_4' f_4'''' + f_0' f_3'''' + f_6' f_2'''' + f_7' f_0'''' + f_8' f_0'''' - (1/2)Sq(5f_8^{v} + \eta f_8^{v}) + 2f_0'' f_8''' + f_1' f_7'''') - \\
& \gamma(2f_0'' f_8''' + 2f_1'' f_7''' + 2f_2'' f_6''' + 2f_3'' f_5''' + 2f_4'' f_4''' + 2f_5'' f_3''' + 2f_6'' f_2''' + 2f_7'' f_1''' + 2f_8'' f_0''' + f_0' f_8'''' + f_1' f_7'''' + f_2' f_6'''' + f_3' f_5'''' + \\
& f_4' f_4'''' + f_0' f_3'''' + f_6' f_2'''' + f_7' f_0'''' + f_8' f_0'''') + \beta(18 f_0' f_1' f_7'''' + 24 f_0' f_0'' f_8''' + 14 f_7'' f_0'' f_1'' + (9(2f_0' f_2' + f_1'^2)) f_6'''' + 7f_2''(2f_0'' f_6'' + \\
& 2f_1'' f_5'' + 2f_2'' f_4'' + f_3''^2) + 7f_3''(2f_0'' f_5'' + 2f_1'' f_4'' + 2f_2'' f_3'') + 7f_4''(2f_0'' f_4'' + 2f_1'' f_3'' + f_2''^2) + \\
& 7f_6''(2f_0'' f_2'' + f_1''^2) + 7f_5''(2f_0'' f_3'' + 2f_1'' f_2'') + 7f_8'' f_0''^2 + (24(f_0' f_5'' + f_1' f_4'' + f_2' f_3'' + f_3' f_2'' + f_4' f_1'' + f_0' f_0'')) f_3''' + \\
& 9f_0'^2 f_8'''' + (24(f_0' f_4'' + f_1' f_3'' + f_2' f_2'' + f_3' f_1'' + f_4' f_0'')) f_4''' + (24(f_0' f_3'' + f_1' f_2'' + f_2' f_1'' + f_3' f_0'')) f_5''' + (24(f_0' f_1'' + f_1' f_0'')) f_7''' + \\
& (24(f_0' f_2'' + f_1' f_1'' + f_2' f_0'')) f_6''' + (9(2f_0' f_3' + 2f_1' f_2')) f_5'''' + (9(2f_0' f_4' + 2f_1' f_3' + f_2'^2)) f_4'''' + (24(f_0' f_6'' + f_1' f_5'' + f_2' f_4'' + f_3' f_3'' + \\
& f_4' f_2'' + f_0' f_1'' + f_6' f_0'')) f_2''' + 7f_0''(2f_0'' f_8'' + 2f_1'' f_7'' + 2f_2'' f_6'' + 2f_3'' f_5'' + f_4''^2) + 7f_1''(2f_0'' f_7'' + 2f_1'' f_6'' + 2f_2'' f_5'' + 2f_3'' f_4'') + \\
& (9(2f_0' f_6' + 2f_1' f_0' + 2f_2' f_4' + f_3'^2)) f_2'''' + (9(2f_0' f_8' + 2f_1' f_7' + 2f_2' f_6' + 2f_3' f_0' + f_4'^2)) f_0'''' + (24(f_0' f_8'' + f_1' f_7'' + f_2' f_6'' + f_3' f_5'' + \\
& (9(2f_0' f_0' + 2f_1' f_4' + 2f_2' f_3')) f_3'''' + (9(2f_0' f_7' + 2f_1' f_6' + f_4' f_4'' + f_0' f_3'' + f_6' f_2'' + f_7' f_1'' + f_8' f_0'')) f_0''' + 2f_2' f_0' + 2f_3' f_4')) f_0'''' + \\
& (24(f_0' f_7'' + f_1' f_6'' + f_2' f_5'' + f_3' f_4'' + f_4' f_3'' + f_0' f_2'' + f_6' f_1'' + f_7' f_0'')) f_1''' + \mathrm{Re}(3f_0''(2f_0''' f_8''' + 2f_1''' f_7''' + 2f_2''' f_6''' + 2f_3''' f_5''' + f_4'''^2) + \\
& 3f_1''(2f_0''' f_7''' + 2f_1''' f_6''' + 2f_2''' f_5''' + 2f_3''' f_4''') + 3f_2''(2f_0''' f_6''' + 2f_1''' f_5''' + 2f_2''' f_4''' + f_3'''^2) + 3f_3''(2f_0''' f_5''' + 2f_1''' f_4''' + 2f_2''' f_3''') + \\
& 3f_4''(2f_0''' f_4''' + 2f_1''' f_3''' + f_2'''^2) + 3f_5''(2f_0''' f_3''' + 2f_1''' f_2''') + 3f_6''(2f_0''' f_2''' + f_1'''^2) + 6f_7'' f_0''' f_1''' + 3f_8'' f_0'''^2 + (3/2) f_0''^2 f_8'''' + 3f_0'' f_1'' f_7'''' + \\
& \frac{3}{2}(2f_0'' f_2'' + f_1''^2)) f_6'''' + (3/2(2f_0'' f_3'' + 2f_1'' f_2'')) f_5'''' + (3/2(2f_0'' f_4'' + 2f_1'' f_3'' + f_2''^2)) f_4'''' + (3/2(2f_0'' f_5'' + 2f_1'' f_4'' + 2f_2'' f_3'')) f_3'''' + \\
& \left(\frac{3}{2}(2f_0'' f_6'' + 2f_1'' f_5'' + 2f_2'' f_4'' + f_3''^2)\right) f_2'''' + (3/2(2f_0'' f_7'' + 2f_1'' f_6'' + 2f_2'' f_5'' + 2f_3'' f_4'')) f_0'''' + \left(3/2\left(\begin{matrix} 2f_0'' f_8'' + 2f_1'' f_7'' + \\ 2f_2'' f_6'' + 2f_3'' f_5'' + f_4''^2 \end{matrix}\right)\right) f_0'''')) = 0
\end{aligned}
$$

## Appendix II

$$
\begin{aligned}
p^5 :\ & (1+Rd)\theta_5'' + \Pr\left(\theta_0' f_4 + \theta_1' f_3 + \theta_2' f_2 + \theta_3' f_1 + \theta_4' f_0 - \frac{Sq}{2}\eta\theta_4'\right) + PrEc(M^2\left(2f_0'f_4' + 2f_1'f_3' + f_2'^2\right) + \left(\frac{6}{\mathrm{Re}}\left(2f_0'f_4' + 2f_1'f_3' + f_2'^2\right)\right) + \\
& 2f_0''f_4'' + 2f_1''f_3'' + f_2''^2 + \alpha((-6f_0'(2f_0'f_4' + 2f_1'f_3' + f_2'^2) - 6f_1'\left(2f_0'f_3' + 2f_1'f_2'\right) - 6f_2'\left(2f_0'f_2' + f_1'^2\right) - 12f_3'f_0'f_1' - 6f_4'f_0'^2 - \\
& 6f_0f_0'f_4'' - \left(6\left(f_0f_1' + f_1f_0'\right)\right)f_3'' - \left(6\left(f_0f_2' + f_1f_1' + f_2f_0'\right)\right)f_2'' - \left(6\left(f_0f_3' + f_1f_2' + f_2f_1' + f_3f_0'\right)\right)f_1'' - \\
& \left(6\left(f_0f_4' + f_1f_3' + f_2f_2' + f_3f_1' + f_4f_0'\right)\right)f_0'')/\mathrm{Re} - f_0'\left(2f_0''f_4'' + 2f_1''f_3'' + f_2''^2\right) - f_1'\left(2f_0''f_3'' + 2f_1''f_2''\right) - f_2'\left(2f_0''f_2'' + f_1''^2\right) - \\
& 2f_3'f_0''f_1'' - f_4'f_0''^2 - f_0f_0''f_4''' - \left(f_0f_1'' + f_1f_0''\right)f_3''' - \left(f_0f_2'' + f_1f_1'' + f_2f_0''\right)f_2''' - \left(f_0f_3'' + f_1f_2'' + f_2f_1'' + f_3f_0''\right)f_1''' - \\
& (f_0f_4'' + f_1f_3'' + f_2f_2'' + f_3f_1'' + f_4f_0'')f_0''' + (1/2)Sq(6f_0''f_4'' + 6f_1''f_3'' + 3f_2''^2 + \eta f_0''f_4''' + \eta f_1''f_3''' + \eta f_2''f_2''' + \eta f_3''f_1''' + \eta f_4''f_0''') + \\
& Sq(12f_0'f_4' + 12f_1'f_3' + 6f_2'^2 + 3\eta f_0'f_4'' + 3\eta f_1'f_3'' + 3\eta f_2'f_2'' + 3\eta f_3'f_1'' + 3\eta f_4'f_0'')/\mathrm{Re} - 3\gamma((2(f_0'\left(2f_0'f_4' + 2f_1'f_3' + f_2'^2\right) + \\
& f_1'\left(2f_0'f_3' + 2f_1'f_2'\right) + f_2'\left(2f_0'f_2' + f_1'^2\right) + 2f_3'f_0'f_1' + f_4'f_0'^2))/\mathrm{Re} + (1/2)f_0'\left(2f_0''f_4'' + 2f_1''f_3'' + f_2''^2\right) + (1/2)f_1'\left(2f_0''f_3'' + 2f_1''f_2''\right) + \\
& (1/2)f_2'\left(2f_0''f_2'' + f_1''^2\right) + f_3'f_0''f_1'' + (1/2)f_4'f_0''^2) + \beta((18(2f_0'^2\left(2f_0'f_4' + 2f_1'f_3' + f_2'^2\right) + 4f_0'f_1'\left(2f_0'f_3' + 2f_1'f_2'\right) + \\
& \left(2f_0'f_2' + f_1'^2\right)^2))/\mathrm{Re} + 6f_0'^2\left(2f_0''f_4'' + 2f_1''f_3'' + f_2''^2\right) + 12f_0'f_1'\left(2f_0''f_3'' + 2f_1''f_2''\right) + \left(6\left(2f_0'f_2' + f_1'^2\right)\right)\left(2f_0''f_2'' + f_1''^2\right) + \\
& (12(2f_0'f_3' + 2f_1'f_2'))f_0''f_1'' + \left(6\left(2f_0'f_4' + 2f_1'f_3' + f_2'^2\right)\right)f_0''^2 + \frac{Re}{2}f_4'''')))=0,
\end{aligned}
$$

$$
\begin{aligned}
p^6 :\ & (1+Rd)\theta_6'' + \Pr\left(\theta_0' f_5 + \theta_1' f_4 + \theta_2' f_3 + \theta_3' f_2 + \theta_4' f_1 + \theta_5' f_0 - (1/2)Sq\eta\theta_5'\right) + PrEc(M^2\left(2f_0'f_5' + 2f_1'f_4' + 2f_2'f_3'\right) + \\
& \left(6\left(2f_0'f_5' + 2f_1'f_4' + 2f_2'f_3'\right)\right)/\mathrm{Re} + 2f_0''f_5'' + 2f_1''f_4'' + 2f_2''f_3'' + \alpha((-6f_0'(2f_0'f_5' + 2f_1'f_4' + 2f_2'f_3') - 6f_1'\left(2f_0'f_4' + 2f_1'f_3' + f_2'^2\right) - \\
& 6f_2'\left(2f_0'f_3' + 2f_1'f_2'\right) - 6f_3'\left(2f_0'f_2' + f_1'^2\right) - 12f_4'f_0'f_1' - 6f_5'f_0'^2 - 6f_0f_0'f_5'' - \left(6\left(f_0f_1' + f_1f_0'\right)\right)f_4'' - \left(6\left(f_0f_2' + f_1f_1' + f_2f_0'\right)\right)f_3'' - \\
& \left(6\left(f_0f_3' + f_1f_2' + f_2f_1' + f_3f_0'\right)\right)f_2'' - \left(6\left(f_0f_4' + f_1f_3' + f_2f_2' + f_3f_1' + f_4f_0'\right)\right)f_1'' - (6(f_0f_5' + f_1f_4' + f_2f_3' + f_3f_2' + f_4f_1' + \\
& f_5f_0'))f_0'')/\mathrm{Re} - f_0'\left(2f_0''f_5'' + 2f_1''f_4'' + 2f_2''f_3''\right) - f_1'\left(2f_0''f_4'' + 2f_1''f_3'' + f_2''^2\right) - f_2'\left(2f_0''f_3'' + 2f_1''f_2''\right) - f_3'\left(2f_0''f_2'' + f_1''^2\right) - \\
& 2f_4'f_0''f_1'' - f_5'f_0''^2 - f_0f_0''f_5''' - \left(f_0f_1'' + f_1f_0''\right)f_4''' - \left(f_0f_2'' + f_1f_1'' + f_2f_0''\right)f_3''' - \left(f_0f_3'' + f_1f_2'' + f_2f_1'' + f_3f_0''\right)f_2''' - \\
& \left(f_0f_4'' + f_1f_3'' + f_2f_2'' + f_3f_1'' + f_4f_0''\right)f_1''' - (f_0f_5'' + f_1f_4'' + f_2f_3'' + f_3f_2'' + f_4f_1'' + f_5f_0'')f_0''' + (1/2)Sq(6f_0''f_5'' + 6f_1''f_4'' + \\
& 6f_2''f_3'' + \eta f_0''f_5''' + \eta f_1''f_4''' + \eta f_2''f_3''' + \eta f_3''f_2''' + \eta f_4''f_1''' + \eta f_5''f_0''') + Sq(12f_0'f_5' + 12f_1'f_4' + 12f_2'f_3' + 3\eta f_0'f_5'' + 3\eta f_1'f_4'' + \\
& 3\eta f_2'f_3'' + 3\eta f_3'f_2'' + 3\eta f_4'f_1'' + 3\eta f_5'f_0'')/\mathrm{Re} - 3\gamma((2(f_0'\left(2f_0'f_5' + 2f_1'f_4' + 2f_2'f_3'\right) + f_1'(2f_0'f_4' + 2f_1'f_3' + f_2'^2) + \\
& f_2'\left(2f_0'f_3' + 2f_1'f_2'\right) + f_3'\left(2f_0'f_2' + f_1'^2\right) + 2f_4'f_0'f_1' + f_5'f_0'^2))/\mathrm{Re} + (1/2)f_0'(2f_0''f_5'' + 2f_1''f_4'' + 2f_2''f_3'') + \\
& f_1'\left(f_0''f_4'' + f_1''f_3'' + \frac{1}{2}f_2''^2\right) + (1/2)f_2'\left(2f_0''f_3'' + 2f_1''f_2''\right) + (1/2)f_3'\left(2f_0''f_2'' + f_1''^2\right) + f_4'f_0''f_1'' + (1/2)f_5'f_0''^2) + \\
& \beta((18(2f_0'^2(2f_0'f_5' + 2f_1'f_4' + 2f_2'f_3') + 4f_0'f_1'\left(2f_0'f_4' + 2f_1'f_3' + f_2'^2\right) + \left(2\left(2f_0'f_2' + f_1'^2\right)\right)\left(2f_0'f_3' + 2f_1'f_2'\right)))/\mathrm{Re} + \\
& 6f_0'^2\left(2f_0''f_5'' + 2f_1''f_4'' + 2f_2''f_3''\right) + 12f_0'f_1'(2f_0''f_4'' + 2f_1''f_3'' + f_2''^2) + \left(6\left(2f_0'f_2' + f_1'^2\right)\right)\left(2f_0''f_3'' + 2f_1''f_2''\right) + \\
& \left(6\left(2f_0'f_3' + 2f_1'f_2'\right)\right)\left(2f_0''f_2'' + f_1''^2\right) + \left(12\left(2f_0'f_4' + 2f_1'f_3' + f_2'^2\right)\right)f_0''f_1'' + (12(f_0'f_5' + f_1'f_4' + f_2'f_3'))f_0''^2 + \frac{Re}{2}f_5''''))) = 0
\end{aligned}
$$

$$
\begin{aligned}
p^{7}:&\,(1+Rd)\,\theta_7''\,\Pr(\theta_0' f_6+\theta_1' f_5+\theta_2' f_4+\theta_3' f_3+\theta_4' f_2+\theta_5' f_1+\theta_6' f_0-(1/2)Sq\eta\theta_6')+PrEc(M^2\left(2f_0'f_6'+2f_1'f_5'+2f_2'f_4'+f_3'^2\right)+\\
&\left(6\left(2f_0'f_6'+2f_1'f_5'+2f_2'f_4'+f_3'^2\right)\right)/\operatorname{Re}+2f_0''f_6''+2f_1''f_5''+2f_2''f_4''+f_3''^2+\alpha((-6f_0'(2f_0'f_6'+2f_1'f_5'+2f_2'f_4'+f_3'^2)-\\
&6f_1'\left(2f_0'f_5'+2f_1'f_4'+2f_2'f_3'\right)-6f_2'\left(2f_0'f_4'+2f_1'f_3'+f_2'^2\right)-6f_3'\left(2f_0'f_3'+2f_1'f_2'\right)-6f_4'\left(2f_0'f_2'+f_1'^2\right)-12f_5'f_0'f_1'-\\
&6f_6'f_0'^2-6f_0f_0'f_6''-\left(6\left(f_0f_1'+f_1f_0'\right)\right)f_5''-\left(6\left(f_0f_2'+f_1f_1'+f_2f_0'\right)\right)f_4''-\left(6\left(f_0f_3'+f_1f_2'+f_2f_1'+f_3f_0'\right)\right)f_3''-\\
&\left(6\left(f_0f_4'+f_1f_3'+f_2f_2'+f_3f_1'+f_4f_0'\right)\right)f_2''-(6(f_0f_5'+f_1f_4'+f_2f_3'+f_3f_2'+f_4f_1'+f_5f_0'))f_1''-(6(f_0f_6'+f_1f_5'+f_2f_4'+\\
&f_3f_3'+f_4f_2'+f_5f_1'+f_6f_0'))f_0'')/\operatorname{Re}-f_0'\left(2f_0''f_6''+2f_1''f_5''+2f_2''f_4''+f_3''^2\right)-f_1'(2f_0''f_5''+2f_1''f_4''+2f_2''f_3'')-\\
&f_2'\left(2f_0''f_4''+2f_1''f_3''+f_2''^2\right)-f_3'(2f_0''f_3''+2f_1''f_2'')-f_4'\left(2f_0''f_2''+f_1''^2\right)-2f_5'f_0''f_1''-f_6'f_0''^2-f_0f_0''f_6'''-\left(f_0f_1''+f_1f_0''\right)f_5'''-\\
&\left(f_0f_2''+f_1f_1''+f_2f_0''\right)f_4'''-\left(f_0f_3''+f_1f_2''+f_2f_1''+f_3f_0''\right)f_3'''-(f_0f_4''+f_1f_3''+f_2f_2''+f_3f_1''+f_4f_0'')f_2'''-\\
&\left(f_0f_5''+f_1f_4''+f_2f_3''+f_3f_2''+f_4f_1''+f_5f_0''\right)f_1'''-\left(f_0f_6''+f_1f_5''+f_2f_4''+f_3f_3''+f_4f_2''+f_5f_1''+f_6f_0''\right)f_0'''+\\
&\frac{Sq}{2}(6f_0''f_6''+6f_1''f_5''+6f_2''f_4''+3f_3''^2+\eta f_0''f_6'''+\eta f_1''f_5'''+\eta f_2''f_4'''+\eta f_3''f_3'''+\eta f_4''f_2'''+\eta f_5''f_1'''+\eta f_6''f_0''')+Sq(12f_0'f_6'+\\
&12f_1'f_5'+12f_2'f_4'+6f_3'^2+3\eta f_0'f_6''+3\eta f_1'f_5''+3\eta f_2'f_4''+3\eta f_3'f_3''+3\eta f_4'f_2''+3\eta f_5'f_1''+3\eta f_6'f_0'')/\operatorname{Re}-\\
&3\gamma((2(f_0'\left(2f_0'f_6'+2f_1'f_5'+2f_2'f_4'+f_3'^2\right)+f_1'\left(2f_0'f_5'+2f_1'f_4'+2f_2'f_3'\right)+f_2'\left(2f_0'f_4'+2f_1'f_3'+f_2'^2\right)+f_3'\left(2f_0'f_3'+2f_1'f_2'\right)+\\
&f_4'\left(2f_0'f_2'+f_1'^2\right)+2f_5'f_0'f_1'+f_6'f_0'^2))/\operatorname{Re}+(1/2)f_0'\left(2f_0''f_6''+2f_1''f_5''+2f_2''f_4''+f_3''^2\right)+(1/2)f_1'\left(2f_0''f_5''+2f_1''f_4''+2f_2''f_3''\right)+\\
&(1/2)f_2'\left(2f_0''f_4''+2f_1''f_3''+f_2''^2\right)+(1/2)f_3'\left(2f_0''f_3''+2f_1''f_2''\right)+(1/2)f_4'\left(2f_0''f_2''+f_1''^2\right)+f_5'f_0''f_1''+(1/2)f_6'f_0''^2)+\\
&\beta((18(2f_0'^2\left(2f_0'f_6'+2f_1'f_5'+2f_2'f_4'+f_3'^2\right)+4f_0'f_1'\left(2f_0'f_5'+2f_1'f_4'+2f_2'f_3'\right)+\left(2\left(2f_0'f_2'+f_1'^2\right)\right)\left(2f_0'f_4'+2f_1'f_3'+f_2'^2\right)+\\
&\left(2f_0'f_3'+2f_1'f_2'\right)^2))/\operatorname{Re}+6f_0'^2\left(2f_0''f_6''+2f_1''f_5''+2f_2''f_4''+f_3''^2\right)+12f_0'f_1'(2f_0''f_5''+2f_1''f_4''+2f_2''f_3'')+\\
&\left(6\left(2f_0'f_2'+f_1'^2\right)\right)\left(2f_0''f_4''+2f_1''f_3''+f_2''^2\right)+\left(6\left(2f_0'f_4'+2f_1'f_3'+f_2'^2\right)\right)\left(2f_0''f_2''+f_1''^2\right)+\left(12\left(2f_0'f_5'+2f_1'f_4'+2f_2'f_3'\right)\right)f_0''f_1''+\\
&\left(6\left(2f_0'f_6'+2f_1'f_5'+2f_2'f_4'+f_3'^2\right)\right)f_0''^2+(1/2)Ref_6'''))) = 0,
\end{aligned}
$$

$$
\begin{aligned}
p^{8}:&\,(1+Rd)\,\theta_8''+\Pr(\theta_0' f_7+\theta_1' f_6+\theta_2' f_5+\theta_3' f_4+\theta_4' f_3+\theta_5' f_2+\theta_6' f_1+\theta_7' f_0-(1/2)Sq\eta\theta_7')+PrEc(M^2\left(2f_0'f_7'+2f_1'f_6'+2f_2'f_5'+2f_3'f_4'\right)+\\
&\left(6\left(2f_0'f_7'+2f_1'f_6'+2f_2'f_5'+2f_3'f_4'\right)\right)/\operatorname{Re}+2f_0''f_7''+2f_1''f_6''+2f_2''f_5''+2f_3''f_4''+\alpha((-6f_0'\left(2f_0'f_7'+2f_1'f_6'+2f_2'f_5'+2f_3'f_4'\right)-\\
&6f_1'\left(2f_0'f_6'+2f_1'f_5'+2f_2'f_4'+f_3'^2\right)-6f_2'\left(2f_0'f_5'+2f_1'f_4'+2f_2'f_3'\right)-6f_3'\left(2f_0'f_4'+2f_1'f_3'+f_2'^2\right)-6f_4'\left(2f_0'f_3'+2f_1'f_2'\right)-6f_5'\left(2f_0'f_2'+f_1'^2\right)-\\
&12f_6'f_0'f_1'-6f_7'f_0'^2-6f_0f_0'f_7''-\left(6\left(f_0f_1'+f_1f_0'\right)\right)f_6''-\left(6\left(f_0f_2'+f_1f_1'+f_2f_0'\right)\right)f_5''-\left(6\left(f_0f_3'+f_1f_2'+f_2f_1'+f_3f_0'\right)\right)f_4''-\\
&\left(6\left(f_0f_4'+f_1f_3'+f_2f_2'+f_3f_1'+f_4f_0'\right)\right)f_3''-\left(6\left(f_0f_5'+f_1f_4'+f_2f_3'+f_3f_2'+f_4f_1'+f_5f_0'\right)\right)f_2''-\left(6\begin{pmatrix}f_0f_6'+f_1f_5'+f_2f_4'+f_3f_3'+\\ f_4f_2'+f_5f_1'+f_6f_0'\end{pmatrix}\right)f_1''-\\
&\left(6\left(f_0f_7'+f_1f_6'+f_2f_5'+f_3f_4'+f_4f_3'+f_5f_2'+f_6f_1'+f_7f_0'\right)\right)f_0'')/\operatorname{Re}-f_0'\left(2f_0''f_7''+2f_1''f_6''+2f_2''f_5''+2f_3''f_4''\right)-f_1'\begin{pmatrix}2f_0''f_6''+2f_1''f_5''+\\ 2f_2''f_4''+f_3''^2\end{pmatrix}-\\
&f_2'\left(2f_0''f_5''+2f_1''f_4''+2f_2''f_3''\right)-f_3'\left(2f_0''f_4''+2f_1''f_3''+f_2''^2\right)-f_4'\left(2f_0''f_3''+2f_1''f_2''\right)-f_5'\left(2f_0''f_2''+f_1''^2\right)-2f_6'f_0''f_1''-f_7'f_0''^2-f_0f_0''f_7'''-\\
&\left(f_0f_1''+f_1f_0''\right)f_6'''-\left(f_0f_2''+f_1f_1''+f_2f_0''\right)f_5'''-\left(f_0f_3''+f_1f_2''+f_2f_1''+f_3f_0''\right)f_4'''-\left(f_0f_4''+f_1f_3''+f_2f_2''+f_3f_1''+f_4f_0''\right)f_3'''-\\
&\left(f_0f_5''+f_1f_4''+f_2f_3''+f_3f_2''+f_4f_1''+f_5f_0''\right)f_2'''-\left(f_0f_6''+f_1f_5''+f_2f_4''+f_3f_3''+f_4f_2''+f_5f_1''+f_6f_0''\right)f_1'''-\begin{pmatrix}f_0f_7''+f_1f_6''+f_2f_5''+f_3f_4''+\\ f_4f_3''+f_5f_2''+f_6f_1''+f_7f_0''\end{pmatrix}f_0'''+\\
&(1/2)Sq(6f_0''f_7''+6f_1''f_6''+6f_2''f_5''+6f_3''f_4''+\eta f_0''f_7'''+\eta f_1''f_6'''+\eta f_2''f_5'''+\eta f_3''f_4'''+\eta f_4''f_3'''+\eta f_5''f_2'''+\eta f_6''f_1'''+\eta f_7''f_0''')+\frac{Sq}{\operatorname{Re}}(12f_0'f_7'+12f_1'f_6'+\\
&12f_2'f_5'+12f_3'f_4'+3\eta f_0'f_7''+3\eta f_1'f_6''+3\eta f_2'f_5''+3\eta f_3'f_4''+3\eta f_4'f_3''+3\eta f_5'f_2''+3\eta f_6'f_1''+3\eta f_7'f_0'')-3\gamma((2(f_0'\left(2f_0'f_7'+2f_1'f_6'+2f_2'f_5'+2f_3'f_4'\right)+\\
&f_1'\left(2f_0'f_6'+2f_1'f_5'+2f_2'f_4'+f_3'^2\right)+f_2'(2f_0'f_5'+2f_1'f_4'+f_4'\left(2f_0'f_3'+2f_1'f_2'\right)+f_5'\left(2f_0'f_2'+f_1'^2\right)+2f_6'f_0'f_1'+f_7'f_0'^2))/\operatorname{Re}+(1/2)f_0'\begin{pmatrix}2f_0''f_7''+2f_1''f_6''+\\ 2f_2''f_5''+2f_3''f_4''\end{pmatrix}+\\
&(1/2)f_1'\left(2f_0''f_6''+2f_1''f_5''+2f_2''f_4''+f_3''^2\right)+(1/2)f_2'\left(2f_0''f_5''+2f_1''f_4''+2f_2''f_3''\right)+(1/2)f_3'\left(2f_0''f_4''+2f_1''f_3''+f_2''^2\right)+(1/2)f_4'\left(2f_0''f_3''+2f_1''f_2''\right)+\\
&(1/2)f_5'(2f_0''f_2''+f_1''^2)+f_6'f_0''f_1''+(1/2)f_7'f_0''^2)+\beta((18(2f_0'^2\left(2f_0'f_7'+2f_1'f_6'+2f_2'f_5'+2f_3'f_4'\right)+4f_0'f_1'\left(2f_0'f_6'+2f_1'f_5'+2f_2'f_4'+f_3'^2\right)+\\
&\left(2\left(2f_0'f_2'+f_1'^2\right)\right)\left(2f_0'f_5'+2f_1'f_4'+2f_2'f_3'\right)+\left(2\left(2f_0'f_3'+2f_1'f_2'\right)\right)\left(2f_0'f_4'+2f_1'f_3'+f_2'^2\right)))/\operatorname{Re}+6f_0'^2\left(2f_0''f_7''+2f_1''f_6''+2f_2''f_5''+2f_3''f_4''\right)+\\
&12f_0'f_1'\left(2f_0''f_6''+2f_1''f_5''+2f_2''f_4''+f_3''^2\right)+\left(6\left(2f_0'f_2'+f_1'^2\right)\right)\left(2f_0''f_5''+2f_1''f_4''+2f_2''f_3''\right)+\left(6\left(2f_0'f_3'+2f_1'f_2'\right)\right)\left(2f_0''f_4''+2f_1''f_3''+f_2''^2\right)+\\
&\left(6\left(2f_0'f_4'+2f_1'f_3'+f_2'^2\right)\right)\left(2f_0''f_3''+2f_1''f_2''\right)+\left(6\left(2f_0'f_5'+2f_1'f_4'+2f_2'f_3'\right)\right)\left(2f_0''f_2''+f_1''^2\right)+\left(12\left(2f_0'f_6'+2f_1'f_5'+2f_2'f_4'+f_3'^2\right)\right)f_0''f_1''+\\
&\left(6\left(2f_0'f_7'+2f_1'f_6'+2f_2'f_5'+2f_3'f_4'\right)\right)f_0''^2+\frac{Re}{2}f_7'''))) = 0,
\end{aligned}
$$

$$
\begin{aligned}
& p^{9} : (1+Rd)\theta''_9 + \Pr(\theta'_0 f_8 + \theta'_1 f_7 + \theta'_2 f_6 + \theta'_3 f_5 + \theta'_4 f_4 + \theta'_5 f_3 + \theta'_6 f_2 + \theta'_7 f_1 + \theta'_8 f_0 - \frac{Sq}{2}\eta\theta'_8) + PrEcM^2\left(2f'_0f'_8 + 2f'_1f'_7 + 2f'_2f'_6 + 2f'_3f'_5 + f'^2_4\right) + \\
& \left(6\left(2f'_0f'_8 + 2f'_1f'_7 + 2f'_2f'_6 + 2f'_3f'_5 + f'^2_4\right)\right)/\mathrm{Re} + 2f''_0f''_8 + 2f''_1f''_7 + 2f''_2f''_6 + 2f''_3f''_5 + f''^2_4 + \alpha((-6f'_0\left(2f'_0f'_8 + 2f'_1f'_7 + 2f'_2f'_6 + 2f'_3f'_5 + f'^2_4\right) - \\
& 6f'_1\left(2f'_0f'_7 + 2f'_1f'_6 + 2f'_2f'_5 + 2f'_3f'_4\right) - 6f'_2\left(2f'_0f'_6 + 2f'_1f'_5 + 2f'_2f'_4 + f'^2_3\right) - 6f'_3\left(2f'_0f'_5 + 2f'_1f'_4 + 2f'_2f'_3\right) - 6f'_4\left(2f'_0f'_4 + 2f'_1f'_3 + f'^2_2\right) - \\
& 6f'_5\left(2f'_0f'_3 + 2f'_1f'_2\right) - 6f'_6\left(2f'_0f'_2 + f'^2_1\right) - 12f'_7f'_0f'_1 - 6f'_8f'^2_0 - 6f_0f'_0f''_8 - \left(6\left(f_0f'_1 + f_1f'_0\right)\right)f''_7 - \left(6\left(f_0f'_2 + f_1f'_1 + f_2f'_0\right)\right)f''_6 - \\
& (6(f_0f'_3 + f_1f'_2 + f_2f'_1 + f_3f'_0))f''_5 - \left(6\left(f_0f'_4 + f_1f'_3 + f_2f'_2 + f_3f'_1 + f_4f'_0\right)\right)f''_4 - \left(6\left(f_0f'_5 + f_1f'_4 + f_2f'_3 + f_3f'_2 + f_4f'_1 + f_5f'_0\right)\right)f''_3 - \\
& \left(6\left(f_0f'_6 + f_1f'_5 + f_2f'_4 + f_3f'_3 + f_4f'_2 + f_5f'_1 + f_6f'_0\right)\right)f''_2 - (6(f_0f'_7 + f_1f'_6 + f_2f'_5 + f_3f'_4 + f_4f'_3 + f_5f'_2 + f_6f'_1 + f_7f'_0))f''_1 - (6(f_0f'_8 + f_1f'_7 + \\
& f_2f'_6 + f_3f'_5 + f_4f'_4 + f_5f'_3 + f_6f'_2 + f_7f'_1 + f_8f'_0))f''_0)/\mathrm{Re} - f'_0\left(2f''_0f''_8 + 2f''_1f''_7 + 2f''_2f''_6 + 2f''_3f''_5 + f''^2_4\right) - f'_1\left(2f''_0f''_7 + 2f''_1f''_6 + 2f''_2f''_5 + 2f''_3f''_4\right) - \\
& f'_2\left(2f''_0f''_6 + 2f''_1f''_5 + 2f''_2f''_4 + f''^2_3\right) - f'_3\left(2f''_0f''_5 + 2f''_1f''_4 + 2f''_2f''_3\right) - f'_4(2f''_0f''_4 + 2f''_1f''_3 + f''^2_2) - f'_5\left(2f''_0f''_3 + 2f''_1f''_2\right) - f'_6\left(2f''_0f''_2 + f''^2_1\right) - \\
& 2f'_7f''_0f''_1 - f'_8f''^2_0 - f_0f''_0f'''_8 - \left(f_0f''_1 + f_1f''_0\right)f'''_7 - \left(f_0f''_2 + f_1f''_1 + f_2f''_0\right)f'''_6 - \left(f_0f''_3 + f_1f''_2 + f_2f''_1 + f_3f''_0\right)f'''_5 - (f_0f''_4 + f_1f''_3 + f_2f''_2 + f_3f''_1 + \\
& f_4f''_0)f'''_4 - \left(f_0f''_5 + f_1f''_4 + f_2f''_3 + f_3f''_2 + f_4f''_1 + f_5f''_0\right)f'''_3 - \left(f_0f''_6 + f_1f''_5 + f_2f''_4 + f_3f''_3 + f_4f''_2 + f_5f''_1 + f_6f''_0\right)f'''_2 - (f_0f''_7 + f_1f''_6 + f_2f''_5 + \\
& f_3f''_4 + f_4f''_3 + f_5f''_2 + f_6f''_1 + f_7f''_0)f'''_1 - (f_0f''_8 + f_1f''_7 + f_2f''_6 + f_3f''_5 + f_4f''_4 + f_5f''_3 + f_6f''_2 + f_7f''_1 + f_8f''_0)f'''_0 + (1/2)Sq(6f''_0f''_8 + 6f''_1f''_7 + \\
& 6f''_2f''_6 + 6f''_3f''_5 + 3f''^2_4 + \eta f''_0f'''_8 + \eta f''_1f'''_7 + \eta f''_2f'''_6 + \eta f''_3f'''_5 + \eta f''_4f'''_4 + \eta f''_5f'''_3 + \eta f''_6f'''_2 + \eta f''_7f'''_1 + \eta f''_8f'''_0) + Sq(12f'_0f'_8 + 12f'_1f'_7 + 12f'_2f'_6 + \\
& 12f'_3f'_5 + 6f'^2_4 + 3\eta f'_0f''_8 + 3\eta f'_1f''_7 + 3\eta f'_2f''_6 + 3\eta f'_3f''_5 + 3\eta f'_4f''_4 + 3\eta f'_5f''_3 + 3\eta f'_6f''_2 + 3\eta f'_7f''_1 + 3\eta f'_8f''_0)/\mathrm{Re} - 3\gamma((2(f'_0\begin{pmatrix} 2f'_0f'_8 + 2f'_1f'_7 + 2f'_2f'_6 + \\ 2f'_3f'_5 + f'^2_4 \end{pmatrix} + \\
& f'_1\left(2f'_0f'_7 + 2f'_1f'_6 + 2f'_2f'_5 + 2f'_3f'_4\right) + f'_2\left(2f'_0f'_6 + 2f'_1f'_5 + 2f'_2f'_4 + f'^2_3\right) + f'_3\left(2f'_0f'_5 + 2f'_1f'_4 + 2f'_2f'_3\right) + f'_4\left(2f'_0f'_4 + 2f'_1f'_3 + f'^2_2\right) + f'_5\left(2f'_0f'_3 + 2f'_1f'_2\right) + \\
& f'_6\left(2f'_0f'_2 + f'^2_1\right) + 2f'_7f'_0f'_1 + f'_8f'^2_0))/\mathrm{Re} + (1/2)f'_0\left(2f''_0f''_8 + 2f''_1f''_7 + 2f''_2f''_6 + 2f''_3f''_5 + f''^2_4\right) + (1/2)f'_1\left(2f''_0f''_7 + 2f''_1f''_6 + 2f''_2f''_5 + 2f''_3f''_4\right) + \\
& (1/2)f'_2\left(2f''_0f''_6 + 2f''_1f''_5 + 2f''_2f''_4 + f''^2_3\right) + (1/2)f'_3\left(2f''_0f''_5 + 2f''_1f''_4 + 2f''_2f''_3\right) + (1/2)f'_4\left(2f''_0f''_4 + 2f''_1f''_3 + f''^2_2\right) + (1/2)f'_5\left(2f''_0f''_3 + 2f''_1f''_2\right) + \\
& (1/2)f'_6\left(2f''_0f''_2 + f''^2_1\right) + f'_7f''_0f''_1 + (1/2)f'_8f''^2_0) + \beta((18(2f'^2_0(2f'_0f'_8 + 2f'_1f'_7 + 2f'_2f'_6 + 2f'_3f'_5 + f'^2_4) + 4f'_0f'_1(2f'_0f'_7 + 2f'_1f'_6 + 2f'_2f'_5 + 2f'_3f'_4) + \\
& \left(2\left(2f'_0f'_2 + f'^2_1\right)\right)\left(2f'_0f'_6 + 2f'_1f'_5 + 2f'_2f'_4 + f'^2_3\right) + \left(2\left(2f'_0f'_3 + 2f'_1f'_2\right)\right)\left(2f'_0f'_5 + 2f'_1f'_4 + 2f'_2f'_3\right) + \left(2f'_0f'_4 + 2f'_1f'_3 + f'^2_2\right)^2))/\mathrm{Re} + 6f'^2_0(2f''_0f''_8 + 2f''_1f''_7 + \\
& 2f''_2f''_6 + 2f''_3f''_5 + f''^2_4) + 12f'_0f'_1(2f''_0f''_7 + 2f''_1f''_6 + 2f''_2f''_5 + 2f''_3f''_4) + \left(6\left(2f'_0f'_2 + f'^2_1\right)\right)\left(2f''_0f''_6 + 2f''_1f''_5 + 2f''_2f''_4 + f''^2_3\right) + \\
& \left(6\left(2f'_0f'_5 + 2f'_1f'_4 + 2f'_2f'_3\right)\right)\left(2f''_0f''_3 + 2f''_1f''_2\right) + \left(6\left(2f'_0f'_3 + 2f'_1f'_2\right)\right)\left(2f''_0f''_5 + 2f''_1f''_4 + 2f''_2f''_3\right) + \left(6\left(2f'_0f'_6 + 2f'_1f'_5 + 2f'_2f'_4 + f'^2_3\right)\right)
\end{aligned}
$$