The Journal of Engineering and Exact Sciences – jCEC, Vol. 08N. 01 (2022)
journal homepage: https://periodicos.ufv.br/ojs/jcec
eISSN: 2527-1075
ISSN: 2446-9416

Analytical Solutions of Black-Scholes Partial Differential Equation of Pricing for the Valuations of Financial Options using Hybrid Transformation Methods

Article Info:
Article history: Received 2022-01-19 / Accepted 2022-06-01 / Available online 2022-06-01
doi: 10.18540/jcecvl8iss1pp15223-01i

Zainab Olabisi Dere
ORCID: https://orcid.org/0000-0002-1965-0551
Department of Mathematics, Florida State University, USA
E-mail: zod20@fsu.edu
Gbeminiyi Musibau Sobamowo
ORCID: https://orcid.org/0000-0003-2402-1423
Department of Mechanical Engineering, Faculty of Engineering, University of Lagos, Akoka Lagos, Nigeria
E-mail: gsobamowo@unilag.edu.ng
Antonio Marcos de Oliveira Siqueira
ORCID: https://orcid.org/0000-0001-9334-0394
Federal University of Viçosa, Brazil
E-mail: antonio.siqueira@ufv.br

Abstract
Black–Scholes partial differential equation is a generally acceptable model in financial markets for option pricing. However, without variable transformations, the provision of symbolic solutions to the variable coefficient partial differential equation is not a straight-forward task. Moreover, the coefficients of the Black–Scholes can depend on the time and the asset price which makes the analytical solution of the Black–Scholes model very difficult to develop. In this paper, analytical solutions of the model of valuations of financial options are presented using Laplace and differential transform methods. The results of the solutions of the Laplace and differential transformation methods are compared with the results of the exact analytical solutions. Moreover, numerical examples for different options pricing are presented to establish the applications, speed and accuracy of the hybrid methods.
Keywords: Black–Scholes model. Partial differential Equation. Financial Market. Option pricing. Laplace-differential transformation method.

1. Introduction

The last few years have witnessed tremendous growth in the financial and economic fields owing to the continuous increase in the number of investors and funds in the investment activities as well as diverse financial derivative products which serves as alternative investments. In such activities, financial derivates act as the investment instruments that are derivatives of a financial asset which value depends on the price of the financial asset such as an option contract.

The options are derivatives of financial securities that give or allow buyers the right, but not the obligation, to buy or sell an underlying asset at a stated (strike) price within a specific timeframe as specified in the contract [1]. An option can be an American option or a European option. An America option is an option which can be exercised at any time until expiration or maturity date while a European option is an option which can be exercised only at a fixed expiration or maturity date [2].

The European option can be of two types, namely, call option and put option. These two options are the basis for a wide range of option strategies that are designed for hedging, income, or speculation. Call options give or allow the buyer or the holder the right, but not the obligation, to buy the underlying asset such as stock, bond, or commodity at a stated (strike) price within a specific timeframe in the contract. Put options give or allow the buyer or the holder the right, but not the obligation, to sell the underlying asset at a stated (strike) price within a specific timeframe in the contract.

Option pricing has been efficiently modelled by the well-known Black–Scholes second-order partial differential equation [3]. The Black-Scholes model can be used for European or American option pricing [4-6]. In order to achieve this, there have been several attempts to produce analytical solutions to the second-order partial differential equation [6]. However, the coefficients of the Black–Scholes can depend on the time and the asset price. Consequently, the analytical solution of the generalized Black–Scholes model is not a straight-forward task.

Therefore, over the years, various numerical methods have been presented to solve the option pricing problems [7–20]. In the computational adventures for the numerical solutions for the Black–Scholes model, Kadalbajoo et al. [21] adopted a method of cubic B-spline collocation for the generalized Black–Scholes model while Valkoy [22] utilized a fitted finite volume method for the same financial model. Huang [23] explored a cubic spline method for the generalized equation of option pricing. In a further work, Mohammadi [24] applied a coupled approach of quintic B-spline collocation method, Euler method and Crank–Nicolson method to solve the second-order PDE for the valuations of financial options.

The various applications of finite difference method depict the fact that the numerical method is an accurate and efficient method for the financial problem. Also, it ensures stability of the scheme for any given volatility and interest rates. Indisputably, it is required that the qualitative properties of the solution are reproduced by the numerical methods, but most option pricing have inherent non-smooth payoffs or discontinuous derivatives at the exercise price. Under such situation, the conventional finite difference schemes with non-smooth payoffs and large time-steps are not efficient because of the presence of the discontinuities in the source terms. Furthermore, in the course of estimating the hedging parameters, the unwanted oscillations become very challenging.

Therefore, in the recent years, different approximate analytical or series solutions methods such as homotopy perturbation method, homotopy analysis method, variational iteration method, projected differential transformation method, Adomian decomposition method [25-32] have been utilized to solve the Black-Scholes pricing model. Such series solutions methods are very essential and necessary especially when there are considerations of more complicated option pricing problems that do not accept symbolic solutions in simple closed forms. However, the series solutions provide a non-smooth analytical solution at a single point, i.e., when the exercise or strike price is equal to the stock price [33]. Consequently, Sumiati et al. [34] developed non-series solutions with partly series solution method for the Black–Scholes second-order partial differential equation using Laplace-Adomian decomposition method.

However, the relatively high computational efforts and time in the determination of Adomian polynomials during the application of Adomian decomposition method has been the major drawback in the approach. In the quest for a relatively simplified approach, the obvious advantages of combining Laplace transform with differential transform method for solving nonlinear differential equations are well established. Indisputably, the various applications of Laplace-differential transformation method have shown that the hybrid method can produce solutions to nonlinear differential without linearization, restrictive or weak nonlinear assumptions, perturbation, discretization.

The hybrid method is very simple in approach and well applicable in solving algebraic, integral, stochastic differential equations, differential and integro-differential equations of integer or fractional order. To the best of the authors' knowledge, such a hybrid transformations method has not been applied to solve Black-Scholes' model. In this paper, an analytical solution for the model of European option is presented using Laplace-differential transformation methods. Also, the partly series solution method is used to develop non-series solutions for the call and put options pricing. The results of the solutions of the Laplace and differential transform methods are compared with the results of the existing exact analytical solution. Moreover, numerical examples for options pricing and parametric studies are presented to establish the applications, speeds and accuracies of the hybrid method.

2. The Black-Scholes Model

The Black-Scholes model is a mathematical model for the valuation of financial options. The derivations of this model can be found in the Appendix. The Black-Scholes second-order partial differential equation is given as [3, 4 and 5].

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \quad (1)$$

where
V is the value of the option which explicitly depends on the current asset price and time.
S is the current price of the underlying asset
r is the interest rate (risk-free rate)
t is time to maturity or expiration
σ is volatility of the underlying asset
The required value $V(S, t)$ will provide us with the information on how much should be paid now, at time t, to hold that option if the current asset price is S.

The above Black-Scholes Model is based on the following assumptions [3]:

i. The stock price V follows the Geometric Brownian Motion with constant drift μ and volatility σ.
ii. The short selling of securities with full use of proceeds is permitted
iii. There are no transactions costs or taxes. All securities are perfectly divisible.
iv. There are no dividends during the life of the option.
v. There are no riskless arbitrage opportunities.
vi. Security trading is continuous.
vii. The risk-free rate of interest, r, is constant and the same for all maturities.

If the option is to buy the asset, it is a call option, C. Ifit is to sell asset, it is a put option, P.
The Black-Scholes equation and boundary conditions for a European call option with values $C(S, t)$ are, as described in.

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0 \quad (2)$$

With the initial condition

$$C(S,0)=max(S-E,0) \quad (3)$$

And the boundary conditions

$$C(0,t)=0\,,\quad C(S,t)\sim S,\quad S\to\infty \tag{4}$$

E, *T* and *r* are the exercise (strike) price and the expiry, respectively

The above partial differential equation with variable coefficient as presented in Eq.(2) can be transformed to partial differential equation with constant coefficient using the following variable transformations:

$$S=Ee^{x}\,,\quad t=T-\frac{\tau}{\frac{1}{2}\sigma^{2}}\,,\quad C=Ev(x,t) \tag{5}$$

Applying Eq. (5) in Eqs. (2), (3) and (4), we have

$$\frac{\partial v}{\partial \tau}=\frac{\partial^{2} v}{\partial x^{2}}+(k-1)\frac{\partial v}{\partial x}-kv \tag{6}$$

where

$$k=\frac{2r}{\sigma^{2}} \tag{7}$$

The initial condition for the European option is

$$v(x,0)=max(e^{x}-1,0) \tag{8}$$

And the boundary conditions

$$v(0,\tau)=0\,,\;\; v(x,\tau)\sim\frac{C}{E},\quad x\to\infty \tag{9}$$

3. Analytical Solutions Black-Scholes Model using Hybrid Transformation Methods

Indisputably, the Black-Scholes model can easily be solved numerically. However, in the analysis of the transient problems, we made recourse to symbolic solutions of the problems. Such symbolic solution will provide better physical insights into the importance of model parameters than the numerical methods. In the generation of the analytical solutions to differential equations, the practical significance of transform methods facilitates observation of great many properties and hidden views, of both mathematical and physical interest, which are not yet well known and have not met with proper appreciation. Consequently, hybridizing Laplace and differential transformation methods, analytical solutions are developed for the Black-Scholes option pricing model as presented in this work.

3.1. Laplace transform method (LT)

The LT of function $\theta(t)$ and corresponding inversion are enumerated as

$$\Theta(s)=\int_{0}^{\infty}e^{-st}\theta(t)dt \tag{10}$$

$$\theta(t)=\frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty}e^{-st}\Theta(s)dt \tag{11}$$

where $s=a+ib$ $(a, b \in R)$ is a complex number.

3.2 Differential Transformation method

Due to the nonlinear terms in Eq. (11), we combined differential transformation method (DTM) to the Laplace transform. This is because of its relative simplicity and advantages for providing the acceptable symbolic results. The definition, basic definitions, principle and the operational properties of the method are given below

3.2.1 The basic definitions and operational properties of DTM

If $u(t)$ is analytic in the domain *T*, then it will be differentiated continuously with respect to time *t*.

$$\frac{d^k u(t)}{dt^k} = \varphi(t,k) \qquad \text{for} \qquad \text{all } t \in T \tag{12}$$

for $t = t_i$, then $\varphi(t,k) = \varphi(t_i,k)$, where *k* belongs to the set of non-negative integers, denoted as the *k*-domain. Therefore Eq. (12) is expressed as

$$U(k) = \varphi(t_i,k) = \left[\frac{d^k u(t)}{dt^k}\right]_{t=t_i} \tag{13}$$

Where U_k is the spectrum of $u(t)$ at $t = t_i$

If $u(t)$ can be expanded by Taylor's series, the $u(t)$ can be written as

$$u(t) = \sum_{k}^{\infty}\left[\frac{(t-t_i)^k}{k!}\right] U(k) \tag{14}$$

$u(t)$ is the inverse of $U(k)$

Eq. (14) can be written as

$$u(t) = \sum_{k=0}^{\infty}\left[\frac{(t-t_i)^k}{k!}\right] U(k) = D^{-1}U(k) \tag{15}$$

where '*D*' denotes the differential transformation process and D^{-1} represents the inverse.

3.2.2. Operational properties of differential transformation method

If $u(t)$ *and* $v(t)$ are two independent functions with time (t) where $U(k)$ and $V(k)$ are the transformed function corresponding to $u(t)$ and $v(t)$, then it can be proved from the fundamental mathematics operations performed by differential transformation that

i. If $z(t) = u(t) \pm v(t)$, then $Z(k) = U(k) \pm V(k)$

ii. If $z(t) = \alpha u(t)$, then $Z(k) = \alpha U(k)$

iii. If $z(t) = \frac{du(t)}{dt}$, then $Z(k) = (k-1)\,U(k+1)$

iv. If $z(t) = u(t)v(t)$, then $Z(t) = \sum_{i=0}^{K} V(l)U(k-l)$

v. If $z(t) = u^{m}(t)$, then $Z(t) = \sum_{I=0}^{K} U^{m-1}(l)U(k-l)$

vi. If $z(t) = u^{n}(t)v^{n}(t)$, then $Z(t) = \sum_{l-0}^{k} \left[\sum_{j=0}^{l} [V(j)U(l-j)] \;\; \sum_{j=0}^{k-l} [V(j)U(k-l-j)] \right]$

vii. If $z(t) = u(t)v(t)$, then $Z(k) = \sum_{l=0}^{k} (l+1)V(l+1)U(k-l)$

viii. If $z(t) = t^{m}$, then $Z(t) = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}$

3.3 Solution Procedures for the Financial Model

Applying Laplace transform to the space domain in Eq. (6), we have

$$\frac{\partial \tilde{v}}{\partial \tau} = s^{2}\tilde{v} - sv_{0} - v_{0}' + (k-1)\left(s\tilde{v} - v_{0}\right) - k\tilde{v} \tag{16}$$

Collecting like terms, we have

$$\frac{\partial \tilde{v}}{\partial \tau} = s^{2}\tilde{v} - v_{0}' + (k-1)s\tilde{v} - k\tilde{v} \tag{17}$$

Which can be written as

$$\frac{\partial \tilde{v}}{\partial \tau} = s^{2}\tilde{v} - v_{0}' + s\left(k-1\right)\tilde{v} - k\tilde{v} \tag{18}$$

Applying differential transform to Eq. (18), we obtain

$$\left(m+1\right)\tilde{V}\left(m+1\right) = s^{2}\tilde{V}(m) - v_{0}'\lambda(m) + ks\tilde{V}(m) - s\tilde{V}(m) - k\tilde{V}(m) \tag{19}$$

Therefore, we have

$$\tilde{V}(m) = \frac{1}{s^{2}}\left[\left(m+1\right)\tilde{V}\left(m+1\right)) + v_{0}'\lambda(m) - ks\tilde{V}(m) + s\tilde{V}(m) + k\tilde{V}(m)\right] \tag{20}$$

where

$$\lambda(m)=\begin{cases}1 & m=0\\ 0 & m\neq 0\end{cases} \tag{21}$$

The inverse Laplace transform From Eq. (20) is

$$V(m)=L^{-1}\left[\frac{1}{s^{2}}\left[\left(m+1\right)\tilde{V}\left(m+1\right)\right)+v_0'\lambda(m)-ks\tilde{V}(m)+s\tilde{V}(m)+k\tilde{V}(m)\right]\right] \tag{22}$$

The initial condition has been given in Eq. (8) as

$$v_0=max(e^{x}-1,0)\rightarrow V\left(0\right)=max(e^{x}-1,0)$$

After straightforward iterative steps and Laplace inversion of Eq. (22), we have

$$V(1)=kmax(e^{x},0)-kmax(e^{x}-1,0); \tag{23}$$

$$V(2)=-\left[\frac{1}{2}k^{2}max(e^{x},0)-\frac{1}{2}k^{2}max(e^{x}-1,0)\right] \tag{24}$$

$$V(3)=\frac{1}{6}k^{3}max(e^{x},0)-\frac{1}{6}k^{3}max(e^{x}-1,0) \tag{25}$$

$$V(4)=-\left[\frac{1}{24}k^{4}max(e^{x},0)-\frac{1}{24}k^{4}max(e^{x}-1,0)\right] \tag{26}$$

Generally,

$$V(n)=\left(-1\right)^{n+1}\left[\frac{1}{n!}k^{n}max(e^{x},0)-\frac{1}{n!}k^{n}max(e^{x}-1,0)\right] \tag{27}$$

For n=1,2,3….

From the definition of differential transformation method

$$v(x,\tau)=V(0)+V(1)\tau+V(2)\tau^{2}+V(3)\tau^{3}+V(4)\tau^{4}+...+V(n)\tau^{n} \tag{28}$$

Therefore, we have

$$\begin{aligned}v(x,\tau)=\;&max(e^{x}-1,0)+\left[kmax(e^{x},0)-kmax(e^{x}-1,0)\right]\tau-\left[\frac{1}{2}k^{2}max(e^{x},0)-\frac{1}{2}k^{2}max(e^{x}-1,0)\right]\tau^{2}\\ &+\left[\frac{1}{6}k^{3}max(e^{x},0)-\frac{1}{6}k^{3}max(e^{x}-1,0)\right]\tau^{3}-\left[\frac{1}{24}k^{4}max(e^{x},0)-\frac{1}{24}k^{4}max(e^{x}-1,0)\right]\tau^{4}\\ &+...+\left(-1\right)^{n+1}\left[\frac{1}{n!}k^{n}max(e^{x},0)-\frac{1}{n!}k^{n}max(e^{x}-1,0)\right]\tau^{n}\end{aligned} \tag{29}$$

Which gives,

$$\begin{aligned} v(x,\tau) &= max(e^x-1,0)+k\tau max(e^x,0)-k\tau max(e^x-1,0)-\frac{1}{2}(k\tau)^2 max(e^x,0)+\frac{1}{2}(k\tau)^2 max(e^x-1,0) \\ &+\frac{1}{6}(k\tau)^3 max(e^x,0)-\frac{1}{6}(k\tau)^3 max(e^x-1,0)-\frac{1}{24}(k\tau)^4 max(e^x,0)+\frac{1}{24}(k\tau)^4 max(e^x-1,0) \\ &+...+(-1)^{n+1}\left[\frac{1}{n!}(k\tau)^n max(e^x,0)-\frac{1}{n!}(k\tau)^n max(e^x-1,0)\right] \end{aligned} \quad (30)$$

We can write that

$$v(x,\tau)=max(e^x-1,0)+\sum_{n=1}^{N}(-1)^{n+1}\left[\frac{1}{n!}(k\tau)^n max(e^x,0)-\frac{1}{n!}(kt)^n max(e^x-1,0)\right] \quad (31)$$

Recall that

$$e^x=\frac{S}{E},\quad t=T-\frac{\tau}{\frac{1}{2}\sigma^2},\quad v(x,t)=\frac{C(x,t)}{E},\quad k=\frac{2r}{\sigma^2}$$

Therefore,

$$C(x,t)=E\left\{max\left(\frac{S}{E}-1,0\right)+\sum_{n=1}^{N}\frac{(-1)^{n+1}\left[r(T-t)\right]^n}{n!}\left[max\left(\frac{S}{E},0\right)-max\left(\frac{S}{E}-1,0\right)\right]\right\} \quad (32)$$

Furthermore, we can the collect like terms in Eq. (30) to have

$$\begin{aligned} v(x,\tau) &= k\tau max(e^x,0)-\frac{1}{2}(k\tau)^2 max(e^x,0)+\frac{1}{6}(k\tau)^3 max(e^x,0)-\frac{1}{24}(k\tau)^4 max(e^x,0) \\ &+(-1)^{n+1}\left[\frac{1}{n!}(k\tau)^n max(e^x,0)\right]+..+max(e^x-1,0)-k\tau max(e^x-1,0)+\frac{1}{2}(k\tau)^2 max(e^x-1,0) \\ &-\frac{1}{6}(k\tau)^3 max(e^x-1,0)+\frac{1}{24}(k\tau)^4 max(e^x-1,0)-(-1)^{n+1}\frac{1}{n!}(k\tau)^n max(e^x-1,0) \end{aligned} \quad (33)$$

After factorization

$$\begin{aligned} v(x,\tau) &= \left\{1-k\tau+\frac{1}{2}(k\tau)^2-\frac{1}{6}(k\tau)^3+\frac{1}{24}(k\tau)^4-...-(-1)^{n+1}\left[\frac{1}{n!}(k\tau)^n\right]\right\}max(e^x-1,0) \\ &+\left\{k\tau-\frac{1}{2}(k\tau)^2+\frac{1}{6}(k\tau)^3-\frac{1}{24}(k\tau)^4+...+(-1)^{n+1}\left[\frac{1}{n!}(k\tau)^n\right]\right\}max(e^x,0) \end{aligned} \quad (34)$$

By series expansion,

$$e^{-k\tau}\approx 1-k\tau+\frac{1}{2}(k\tau)^2-\frac{1}{6}(k\tau)^3+\frac{1}{24}(k\tau)^4-...-(-1)^{n+1}\left[\frac{1}{n!}(k\tau)^n\right] \quad (35)$$

$$1-e^{-k\tau}\approx k\tau-\frac{1}{2}(k\tau)^2+\frac{1}{6}(k\tau)^3-\frac{1}{24}(k\tau)^4+...+(-1)^{n+1}\left[\frac{1}{n!}(k\tau)^n\right] \quad (36)$$

Therefore, the above Eq. (35) can be expressed as,

$$v(x,\tau)=max(e^{x}-1,0)e^{-k\tau}+max(e^{x},0)\left(1-e^{-k\tau}\right) \tag{37}$$

Recall that

$$e^{x}=\frac{S}{E},\quad t=T-\frac{\tau}{\frac{1}{2}\sigma^{2}},\quad v(x,t)=\frac{C(x,t)}{E},\quad k=\frac{2r}{\sigma^{2}},\qquad \tau=\frac{1}{2}\sigma^{2}\left(T-t\right)\rightarrow k\tau=r\left(T-t\right)$$

Therefore, Eq. (37) becomes

$$C(x,t)=E\left\{max\left(\frac{S}{E}-1,0\right)e^{-r(T-t)}+max\left(\frac{S}{E},0\right)\left(1-e^{-r(T-t)}\right)\right\} \tag{38}$$

In conclusion, the call and put option price models are given

$$C(x,t)=E\left\{max\left(\frac{S}{E}-1,0\right)e^{-r(T-t)}+max\left(\frac{S}{E},0\right)\left(1-e^{-r(T-t)}\right)\right\} \tag{39}$$

and

$$P(x,t)=Ee^{-r(T-t)}-S+E\left\{max\left(\frac{S}{E}-1,0\right)e^{-r(T-t)}+max\left(\frac{S}{E},0\right)\left(1-e^{-r(T-t)}\right)\right\} \tag{40}$$

The accuracy of the finite difference solutions can be tested when the results is verified with the results of the analytical solution which is obtained by Fourier transformation. The analytical solution is given as

$$C(S,t)=SN(d_{1})-Ee^{-r(T-t)}N(d_{2}) \tag{41}$$

and

$$P(S,t)=Ee^{-r(T-t)}N(-d_{2})-SN(-d_{1}) \tag{42}$$

where

$$d_{1}=\frac{\mathrm{In}\left(\frac{S}{K}\right)+\frac{1}{2}\sigma^{2}\left(T-t\right)\left(\frac{r}{\frac{1}{2}\sigma^{2}}+1\right)}{\sigma\sqrt{T-t}},\quad d_{2}=\frac{\mathrm{In}\left(\frac{S}{K}\right)+\frac{1}{2}\sigma^{2}\left(T-t\right)\left(\frac{r}{\frac{1}{2}\sigma^{2}}-1\right)}{\sigma\sqrt{T-t}}, \tag{43}$$

On substituting Eqs. (43) into Eqs. (41) and (42), we have

$$C(S,t)=SN\left(\frac{\mathrm{In}\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^{2}\right)\left(T-t\right)}{\sigma\sqrt{T-t}}\right)-Ee^{r(T-t)}N\left(\frac{\mathrm{In}\left(\frac{S}{K}\right)+\left(r-\frac{1}{2}\sigma^{2}\right)\left(T-t\right)}{\sigma\sqrt{T-t}}\right) \tag{44}$$

and

$$P(S,t) = Ee^{-r(T-t)}N\left(-\frac{\text{In}\left(\frac{S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \right) - \text{S}\,N\left(-\frac{\text{In}\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \right) \quad (45)$$

where

$$N(\eta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\eta} e^{-\frac{1}{2}S^2}\,dS$$

T - t is the time remaining till expiration as at time *t*; *N(η)* is the cumulative normal density function.

4. Numerical Examples and Parametric Studies

We consider the valuation of European call option with 1 year to expiration. The stock price is given *S = (21.74, 26.62, 30.56, 34.90, 39.38)*, the exercise/strike price of *K = 0.8,* the risk-neutral interest rate is 0.03 per year, and the volatility is 0.25 per year. Therefore, we have the following: *S = (21.74, 26.62, 30.56, 34.90, 39.38)*, K=0.8, *T = 1, r = 0.03, σ = 0.25.*

Table 1 shows the comparison of results of the Laplace and differential transformation methods and that of the existing exact analytical solution. It could be seen that the results are in excellent agreements. However, the results of the Laplace and differential transformation methods are obtained with less computations and converge faster than that of the exact solutions. This attests to the efficiency and accuracy of the Laplace and differential transformation methods.

Table 1: Comparison of the results when K=0.8, *T = 1, r = 0.03, σ = 0.25.*

Stock Price	Call Option Price (Exact Solution)	Call Option Price (LT-DTM)
21.74	20.96	20.96
26.62	25.84	25.84
30.56	29.78	29.78
34.90	34.12	34.12
39.38	39.60	39.60

Moreover, Table 2 presents the comparisons of the CPU time of the exact analytical method and the hybrid method. It can be stated that the hybrid method applied in this work is faster and posses low computational time than that of the exact analytical method. In order to know the effect of the changing call option price on the stock price. Table 3 presents different call option prices case where after one month, the price of the same call option now trades at $15.04, 20.08, 25.32, 32.84 and 37.96 with expiry time of two months.

Table 2: Comparison of CPU time

	CPU time (s)	
σ	Exact Solution	LT-DTM
0.1	1.65	0.08647
0.2	1.73	0.09898

Table 3: Comparison of the results when K=95, *T = 2/12, r = 0.01, σ = 0.5.*

Option Price	Stock Price (Exact Solution)	Stock Price (LT-DTM)
15.40	106.64	106.16
20.08	112.56	112.56
25.32	118.68	118.68
32.84	126.91	126.91
37.96	132.31	132.31

The combined effects of the simultaneous variations of the strike and stock prices on the call option price is presented in Table 4 when the expiry time is two months, the interest rate (risk-free rate) is 0.01 and the volatility of the underlying asset is 0.5.

Table 4: Comparison of the results when $T = 2/12$, $r = 0.01$, $\sigma = 0.5$.

Strike Price	Stock Price	Option Price (Exact Solution)	Option Price (LT-DTM)
75.00	88.32	15.40	15.40
80.00	94.56	16.72	16.72
85.00	101.19	18.36	18.36
90.00	106.78	19.14	19.14
95.00	113.17	20.58	20.58

5. Conclusion

In this study, for the first time, hybrid method of Laplace and differential transformation methods have been successfully applied to the Black–Scholes Equation for valuations of financial options. The partly series solution method have been used to develop non-series solutions for the call and put options pricing. Numerical examples for the European option valuations were presented. The efficiency, accuracy and speed of the Laplace and differential transformation methods were shown when the results of the new method were compared with the results of the existing exact analytical solutions. It was established that the results obtained converge faster to their associated exact solutions. The method displayed low cost of computation. Therefore, it could be stated that the LT-DTM is very efficient, reliable, and faster in application. The hybrid method can produce highly accurate solutions to nonlinear differential without linearization, restrictive or weak nonlinear assumptions, perturbation, discretization, The method is well applicable in solving algebraic, integral, stochastic differential equations differential and integro-differential equations of integer or fractional order. Hence, it is strongly recommended for solving differential equations in financial mathematics, applied sciences and engineering.

Acknowledgements

The authors appreciate the Florida State University, USA and University of Lagos, Nigeria for provision of enabling environments for the research work.

References

1. Hull. J. C. (2015-2016). Option, Futures and Other Derivatives, Global edition, Eighth edition, Pearson Education Limited.
2. Mardianto, L., Pratama, A. P, Soemarsono, A. R., Hakam, A., Rokhmati, E., Putri, M. (2019) . Comparison of Numerical Methods on Pricing of European Put Options. International Journal of Computing Science and Applied Mathematics. ,5(1).
3. Black, F.; Scholes, M.S. (1973). The pricing of options and corporate liabilities. J. Political Econ. 81, 637–654.
4.. Merton, R. C. (1973). Theory of rational option pricing, Bell J. Econ. Manage. Sci. 4, 141–183.
5. Black, F., M. Scholes, M. (1973) The pricing of options and corporate liabilities, J. Pol. Econ. 81, 637–659.
6. Barles, G. Soner, H. M. (1998). Option pricing with transaction costs and a nonlinear Black–Scholes equation, Finance Stoch. 2, 369–397.
7. Company, R., Jódar, L., Pintos, J. R. (2010). Numerical analysis and computing for option pricing models in illiquid markets, Math. Comput. Modelling, 52 (7–8), 1066–1073.
8. Wilmott, P., Howison, S., Dewynne, J. (1995). The Mathematics of Financial Derivatives: A Student Introduction, Cambridge University Press, Cambridge.

9. Cen, Z.; Le, A. (2011). A robust and accurate finite difference method for a generalized Black-Scholes equation. J. Comput. Appl. Math. 235, 3728–3733.
10. Cho, C.; Kim, T.; Kwon, Y. (2005). Estimation of local volatilities in a generalized Black-Scholes model. Appl. Math. Comput. 162, 1135–1149.
11. Kangro, R.; Nicolaides, R. (2000). Far field boundary conditions for Black-Scholes equations. SIAM J. Numer. Anal. 38, 1357–1368.
12. Ashyralyev, A.; Erdogan, A.S.; Tekalan, S.N. (2019). An investigation on finite difference method for the first order partial differential equation with the nonlocal boundary condition. Appl. Comput. Math. 18, 247–260.
13. Anguelov, R.; Lubuma, J.M. (2001). Contributions to the mathematics of the nonstandard finite difference method and applications. Numer. Methods Partial Differ. Equ. 17, 518–543.
14. Mickens, R.E. (1989). Exact solutions to a finite difference model of a nonlinear reaction-advection equation: Implications for numerical analysis. Numer. Methods Part. Differ. Equ. 5, 313–325.
15. Mickens, R.E. (1994). Nonstandard Finite Difference Models of Differential Equations; World Scientific: Singapore.
16. Khalsaraei, M. M.; Shokri, A.; Mohammadnia, Z.; Sedighi, H.M. (2021). Qualitatively Stable Nonstandard Finite Difference Scheme for Numerical Solution of the Nonlinear Black–Scholes Equation. J. Math. 6679484.
17. Mickens, R.E. (2003). A nonstandard finite difference scheme for a Fisher PDE having nonlinear diffusion. Comput. Math. Appl. 45, 429–436.
18. Mickens, R.E. (2000). Nonstandard finite difference schemes for reaction diffusion equations having linear advection. Numer. Methods Partial Differ. Equ. 4, 361–364.
19. Mickens, R.E.; Jordan, P.M. (2005). A new positivity-preserving nonstandard finite difference scheme for the DWE. Numer. Methods Partial Differ. Equ. 21, 976–985.
20. Cen, Z.; Le, A.; Xu, A. (2012). Exponential time integration and second-order difference scheme for a generalized Black-Scholes equation. J. Appl. Math. Art. 796814.
21. Kadalbajoo, M.K.; Tripathi, L.P.; Kumar, A. (2012). A cubic B-spline collocation method for a numerical solution of the generalized Black-Scholes equation. Math. Comput. Model. 55, 1483–1505.
22. Valkov, R. (2014). Fitted finite volume method for a generalized Black-Scholes equation transformed on finite interval. Numer. Algorithms. 65, 195–220.
23. Huang, J.; Cen, Z. (2014). Cubic spline method for a generalized Black-Scholes equation. Math. Probl. Eng. Art. 484362.
24. Mohammadi, R. (2015). Quintic B-spline collocation approach for solving generalized Black-Scholes equation governing option pricing. Comput. Math. Appl. 69, 777–794.
25. G¨ulkac¸ V. (2010). The homotopy perturbation method for the Black – Scholes equation," J. Stat. Comput. Simul. 80(12), 1349-1354.
26. Kumar, S., Yildirim, A., Khan, Y., Jafari, H., Sayevand, K. and I. Wei, I. (2012). Analytical Solution of Fractional Black-Scholes European Option Pricing Equation by Using Laplace Transform," Journal of Fractional Calculus and Applications. 2(8), 1-9.
27. Allahviranloo, T. and S. S. Behzadi, S.S. (2013). The use of iterative methods for solving Black-Scholes equation," Int. J. Ind. Math. 5(1), 1-11.
28. Edeki, S. O. Ugbebor, O.O and Owoloka, E.A. (2015). Analytical Solutions of the Black–Scholes Pricing Model for European Option Valuation via a Projected Differential Transformation Method," Entropy. 17, 7510-7521.
29. Edeki, S. O. Ugbebor, O.O and Ogundile, O.O. (2019). Analytical Solutions of a Continuous Arithmetic Asian Model for Option Pricing using Projected Differential Transform Method," Engineering Letters. 27(2), 303-310.
30. Biazar, J. and F. Goldoust, F. (2013). The Adomian Decomposition Method for the Black-Scholes Equation," 3rd Int. Conf. Appl. Math. Pharm. Scincences, Singapore, 321-323.

31. Gonz´alez-Gaxiola, O., Ru´ız de Ch´avez J. and Santiago, J.A. (2016). A Nonlinear Option Pricing Model Through the Adomian Decomposition Method," Int. J. Appl. Comput. Math. 2, 435-467.
32. Yavuz, M. and Ozdemir, N. (2018). A Quantitative Approach to Fractional Option Pricing Problems with Decomposition Series," Konuralp Journal of Mathematics, 6(1), 102-109.
33. Putri, E. R. M., Mardianto, L., Hakam, A., Imron, C., Susanto, H. (2021). Removing non-smoothness in solving Black-Scholes equation using aperturbation method. Physics Letters A 402, 127367
34. Sumiati, I., Rusyaman, E. and Sukono. (2019). Black-Scholes Equation Solution Using Laplace-Adomian Decomposition Method. IAENG International Journal of Computer Science, 46(4), 21.

Appendix

Derivation of the Black-Scholes Model

The developed Black-Scholes Model which is also called Black-Scholes-Merton Model is derived in this section.

Taking the underlying asset to follow the geometric Brownian motion,

$$\frac{dS}{S} = \mu dt + \sigma dW \tag{A1}$$

Which gives

$$dS = \mu S dt + \sigma S dW \tag{A2}$$

where

S is the stock price or current price of the underlying asset
dW is the Weiner process
t is time to maturity or expiration
µ stock price rate expectation (the expected rate of return on the stock)
σ is volatility of the underlying asset (stock price movement level)

Given that V is the value of the option which explicitly depends on the current asset price (stock price) and time. With the aid of Ito lemma, we have

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\left(\sigma S\right)^2 dt \tag{A3}$$

On substituting Eq. (1.10) into Eq. (1.11), we have

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S}\left(\mu S dt + \sigma S dW\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\left(\sigma S\right)^2 dt \tag{A4}$$

Let a portfolio consists of one option contract and a number of stocks such that the value of the portfolio can be defined as

$$\Pi = V - \Delta S \tag{A5}$$

where

$$\Delta = \frac{\partial V}{\partial S} \tag{A6}$$

The change in the value of the portfolio is given as

$$d\Pi = dV - \Delta dS \tag{A7}$$

On substituting Eq. (A4) into Eq. (A7), we have

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\left(\mu S dt + \sigma S dW\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt - \Delta dS \tag{A8}$$

The change in the value of the portfolio as a result of risk-free (riskless) interest rate, r, can be written as

$$d\Pi = r\Pi dt \tag{A9}$$

Substituting Eq. (A8) into Eq. (A9), we have

$$\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\left(\mu S dt + \sigma S dW\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt - \Delta dS = r\Pi dt \tag{A10}$$

Which is further simplifies to

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\left(\mu S + \sigma S\frac{dW}{dt}\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 - \Delta\frac{dS}{dt} = r\Pi \tag{A11}$$

Recall from Eq. (A3) that

$$\Pi = V - \frac{\partial V}{\partial S}S \tag{A12}$$

The substitution of Eq. (A12) into Eq. (A11) results into

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\left(\mu S + \sigma S\frac{dW}{dt}\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 - \Delta\frac{dS}{dt} = r\left(V - \frac{\partial V}{\partial S}S\right) \tag{A13}$$

Which can be re-arranged and written as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = \Delta\frac{dS}{dt} - \left(\mu S + \sigma S\frac{dW}{dt}\right)\frac{\partial V}{\partial S} \tag{A14}$$

In Eq. (A6), $\Delta = \frac{\partial V}{\partial S}$

Therefore, Eq. (A14) can be written as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = \frac{\partial V}{\partial S}\frac{dS}{dt} - \left(\mu S + \sigma S\frac{dW}{dt}\right)\frac{\partial V}{\partial S} \tag{A15}$$

Which can be further expressed as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = \left[\frac{dS}{dt} - \left(\mu S + \sigma S\frac{dW}{dt}\right)\right]\frac{\partial V}{\partial S} \tag{A16}$$

From Eq. (A2), one can derive that

$$\frac{dS}{dt} = \mu S + \sigma S \frac{dW}{dt} \qquad \text{(A17)}$$

Which implies that

$$\frac{dS}{dt} - \left(\mu S + \sigma S \frac{dW}{dt} \right) = 0 \qquad \text{(A18)}$$

This reveals that the expression in the block bracket of Eq. (A16) is zero. Which makes the RHS of Eq. (A16) is zero. Therefore, we have the Black-Scholes Model as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad \text{(A19)}$$

where C is Price of a call option, S is stock price or price of underlying asset, K is the strike/exercise price, r is the riskless rate, T is time to maturity, is variance of underlying asset, is standard deviation of the (generally referred to as volatility) underlying asset and is the cumulative normal distribution.